Economics 8108
Macroeconomic Theory
Recitation 1

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Adapted From Erzo Luttmer's Notes

1 Limit of Discrete Euler Equation

Consider a deterministic growth model. The discrete time Euler Equation that we know and love is

$$Du(c_t) = \beta R_{t+1} Du(c_{t+1})$$

Let $R_{t+1} = 1 + r_t\Delta$ and $\beta = e^{-\rho\Delta}$, where Δ is the size of the time step. Also, suppose that the time derivative of consumption is well defined, i.e.,

$$\lim_{\Delta\to 0} \frac{c_{t+\Delta} - c_t}{\Delta} = Dc_t$$

Our Euler equation can be written

$$\begin{aligned}
Du(c_t) &= e^{-\rho\Delta}(1 + r_t\Delta)Du(c_{t+\Delta}) \\
e^{\rho\Delta} &= (1 + r_t\Delta)\frac{Du(c_{t+\Delta})}{Du(c_t)} \\
e^{\rho\Delta} - 1 &= (1 + r_t\Delta)\big[1 + \frac{Du(c_{t+\Delta}) - Du(c_t)}{Du(c_t)}\big] - 1 \\
e^{\rho\Delta} - 1 &= r_t\Delta + \frac{Du(c_{t+\Delta}) - Du(c_t)}{Du(c_t)} + r_t\Delta\frac{Du(c_{t+\Delta}) - Du(c_t)}{Du(c_t)} \\
\frac{e^{\rho\Delta} - 1}{\Delta} &= r_t + \frac{Du(c_{t+\Delta}) - Du(c_t)}{Du(c_t)\Delta} + r_t\frac{Du(c_{t+\Delta}) - Du(c_t)}{Du(c_t)}
\end{aligned}$$

Now, we take the limit of both sides as Δ goes to 0, and we get

$$\rho = r_t + \frac{D^2 u(c_t) Dc_t}{Du(c_t)}$$

Notice that if we had left in lagrange multipliers instead of marginal utility, we would have had

$$\frac{e^{\rho\Delta} - 1}{\Delta} = r_t + \frac{\lambda_{t+\Delta} - \lambda_t}{\lambda_t \Delta} + r_t \frac{\lambda_{t+\Delta} - \lambda_t}{\lambda_t}$$

which gives us

$$\rho = r_t + \frac{D\lambda_t}{\lambda_t}$$

2 Solution to Problem 1

In problem one, we are given a discrete time economy, with periods of length Δ, which is given by the consumption process

$$C_{(n_1)\Delta} = C_{n\Delta} e^{\mu\Delta} \times \begin{cases} 1 & \text{w.p. } 1 - \alpha\Delta \\ \delta & \text{w.p. } \alpha\Delta \end{cases}$$

Preferences are given by

$$E_0 \Big[\sum_{n=0}^{\infty} e^{-\rho\Delta n} \Big(\frac{C_{n\Delta}^{1-\gamma} - 1}{1-\gamma} \Big) \Big]$$

The first task is to price a one period bond. We know that we can find this from the Euler Equation which is

$$\begin{aligned} C_{n\Delta}^{-\gamma} &= e^{-\rho\Delta} R E[C_{(n+1)\Delta}^{-\gamma}] \\ \frac{1}{R} &= e^{-\rho\Delta} E\Big[\Big(\frac{C_{(n+1)\Delta}}{C_{n\Delta}}\Big)^{-\gamma}\Big] \end{aligned}$$

Using the consumption process, we can calculate this expectation as

$$\begin{aligned} q_t &= e^{-\rho\Delta}[(1 - \alpha\Delta) e^{-\gamma\mu\Delta} + \alpha\Delta\delta^{-\gamma} e^{-\gamma\mu\Delta}] \\ q_t &= e^{-\rho\Delta}[e^{-\gamma\mu\Delta} + \alpha\Delta(\delta^{-\gamma} - 1) e^{-\gamma\mu\Delta}] \end{aligned}$$

where $t = n\Delta$. If we want to find the limit of the return to this bond, we can evaluate

$$
\begin{aligned}
r_t &= \lim_{\Delta \to 0} \frac{1 - q_t}{\Delta} \\
r_t &= \lim_{\Delta \to 0} \frac{1 - e^{-(\rho+\gamma\mu)\Delta} - \alpha\Delta(\delta^{-\gamma} - 1)e^{-(\rho+\gamma\mu)\Delta}}{\Delta} \\
r_t &= \lim_{\Delta \to 0} \frac{1 - e^{-(\rho+\gamma\mu)\Delta}}{\Delta} - \alpha(\delta^{-\gamma} - 1)e^{-(\rho+\gamma\mu)\Delta} \\
r_t &= \rho + \gamma\mu - \alpha(\delta^{-\gamma} - 1)
\end{aligned}
$$

You are also asked to find the limit of consumption growth. Note that

$$
\begin{aligned}
\frac{1}{\Delta} E[\frac{C_{(n+1)\Delta} - C_{n\Delta}}{C_{n\Delta}}] &= \frac{e^{\mu\Delta} + \alpha\Delta(\delta - 1)e^{\mu\Delta} - 1}{\Delta} \\
&= \frac{e^{\mu\Delta} - 1}{\Delta} + \alpha(\delta - 1)e^{\mu\Delta} - 1
\end{aligned}
$$

Using a similar identity, the limit goes to

$$
g_t = \mu - \alpha(1 - \delta)
$$

Now note that

$$
\begin{aligned}
r_t &= \rho + \gamma\mu - \alpha(\delta^{-\gamma} - 1) \\
r_t &= \rho + \gamma(\mu - \alpha(1 - \delta)) - \alpha(\delta^{-\gamma} - 1 - \gamma(1 - \delta)) \\
r_t &= \rho + \gamma g_t - \alpha(\delta^{-\gamma} - 1 - \gamma(1 - \delta))
\end{aligned}
$$

The convexity of $\delta^{-\gamma}$ implies that

$$
\begin{aligned}
\delta^{-\gamma} &> 1 + \gamma(1 - \delta) \\
r_t &< \rho + \gamma g_t
\end{aligned}
$$

3 One Sector Growth Model

The Cass Koopmans one-sector growth model is the continuous time analog to the basic discrete time growth models that we have seen many times so far in this course. The simplicity and familiarity of the model makes it relatively easy to recognize and interpret the equilibrium conditions. However, future models that we study this mini will be more complicated and less intuitive. This is a good opportunity to break down the equilibrium conditions and look closely to where they come from in the model. The equilibrium conditions are

$$Du(c_t) = \lambda_t \tag{1}$$

$$r_t = \rho - \frac{D\lambda_t}{\lambda_t} \tag{2}$$

$$v_t = D_1 F(k_t, l_t) \tag{3}$$

$$w_t = D_2 F(k_t, l_t) \tag{4}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda_t a_t = 0 \tag{5}$$

$$q_t r_t = v_t + Dq_t - \delta q_t \tag{6}$$

$$a_t = q_t k_t \tag{7}$$

$$Dk_t = x_t - \delta k_t \tag{8}$$

$$c_t + x_t = F(k_t, l_t) \tag{9}$$

Many of these should be familiar but some may be new. Equation one is the first order condition with respect to consumption and equation two is similar to the first order condition with respect to assets (though we do not derive it this way). Combining (1) and (2) gets the first order condition. Equations (3) and (4) are the first order conditions of the firm. Equation (5) is the transversality condition. Equations (6) and (7) are necessary to relate the capital of the firms to the assets of the household. Equation (8) is the law of motion for capital and Equation (9) is the typical resource constraint.

Now lets look at a model of equilibrium that delivers to us these conditions. The household problem is to solve

$$\max_{a_t, c_t, l_t} \int_0^\infty e^{-\rho t} u(c_t) d_t$$

s.t.

$$Da_t = r_t a_t + w_t l_t - c_t \qquad (\lambda_t e^{-\rho t})$$

$$\liminf_{T\to\infty} \exp\left(-\int_T^t r_s d_s\right) a_T \geq 0$$

non-negativity constraints

Equation (1) can be easily seen to be the first order condition with respect to consumption. As usual, we can get equation (2) from the first order condition with respect to assets. It takes a bit more work. The lagrangian is

$$\mathbb{L} = \int_0^\infty e^{-\rho t}\big[u(c_t) + \lambda_t (r_t a_t + w_t l_t - c_t - Da_t)\big] dt$$

From integration by parts, we have

$$\int_0^\infty e^{-\rho t} \lambda_t D a_t dt = \lim_{T\to\infty} e^{-\rho T}\lambda_T a_T - \lambda_0 a_0 - \int_0^\infty e^{-\rho t}[-\rho\lambda_t + D\lambda_t] a_t dt$$

Thus,

$$\mathbb{L} = \int_0^\infty e^{-\rho t}\big[u(c_t) + \lambda_t(r_t a_t + w_t l_t - c_t - \rho a_t) + D\lambda_t a_t\big]dt$$

Now, the first order condition with respect to a_t is

$$\begin{aligned} \lambda_t r_t - \lambda_t \rho_t + D\lambda_t &= 0 \\ r_t &= \rho_t - \frac{D\lambda_t}{\lambda_t} \end{aligned}$$

This problem could also be written with the present value budget constraint.

$$\int_0^\infty \exp(-\int_0^t r_s d_s) c_t d_t \leq a_0 + \int_0^\infty \exp(-\int_0^t r_s d_s) w_t d_t$$

The second equation can also be derived by taking the first order condition for consumption using the present value budget constraint, and then differentiating the first order condition with respect to time.

$$\begin{aligned} \lambda_0 \exp(-\int_0^t r_s d_s) &= e^{-\rho t} Du(c_t) \\ &= e^{-\rho t}\lambda_t \\ -\lambda_0 \exp(-\int_0^t r_s d_s) r_t &= -\rho e^{-\rho t}\lambda_t + e^{-\rho t} D\lambda_t \\ -e^{-\rho t}\lambda_t r_t &= -\rho e^{-\rho t}\lambda_t + e^{-\rho t} D\lambda_t \\ -r_t &= -\rho + \frac{D\lambda_t}{\lambda_t} \\ r_t &= \rho - \frac{D\lambda_t}{\lambda_t} \end{aligned}$$

The firm problem is

$$\max_{k_t, l_t} F(k_t, l_t) - v_t k_t - w_t l_t$$
$$\text{non-negativity}$$

The factor prices for capital and labor come out of this maximization in the typical manner. Notice that the firm is using capital but the household is saving assets. If we want a model where we think about the risk-free rate on assets and the rental price of capital separately (which will be more important in the two-sector model), we need to account for this in our definition of the equilibrium. In reality, these are going to be the same. But to model it, imagine that there is a bank that holds on to the assets of the consumer, pays the consumer's interest, and invests the assets into a capital stock which it then rents to the firm. This problem looks something like

$$\max_{k_t, x_t} v_t k_t + D a_t - q_t x_t - r_t a_t$$
$$D k_t = x_t - \delta k_t$$
$$a_t = q_t k_t$$

The terms of this equation are revenue from renting capital, change in deposits, investment, and paying interest on deposits. If we substitute the motion of capital and the asset-capital value identity, we get

$$\max_{k_t} v_t k_t + D[q_t k_t] - q_t(D k_t + \delta k_t) - r_t q_t k_t$$
$$\max_{k_t} v_t k_t + q_t D k_t + k_t D q_t - q_t D k_t - q_t \delta k_t - r_t q_t k_t$$
$$\max_{k_t} k_t \left(v_t + D q_t - q_t \delta - r_t q_t \right)$$

And now it is evident that equation (6) must hold. The market clearing condition is the standard: consumption and investment must sum to production.