Economics 8107
Macroeconomic Theory
Recitation 5

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Adapted From Manuel Amador's Notes and LS

Two-Sided Lack of Commitment

This section presents a version of Kocherlakota (1996), following LS approach to 2 risk averse households.

1 Reursive Formulation

The Pareto frontier problem can be rewritten in a recursive way as

$$
\begin{aligned}
(\mathbf{P}) \equiv Q(\Delta, s) = \max_{c,\Delta(s')} & \left\{ u(1-c) - u(1-y(s)) + \beta \sum_{s'} \pi(s) Q(\Delta(s'), s') \right\} \\
s.t. \quad & u(c) - u(y(s)) + \beta \sum_{s'} \pi(s')\Delta(s') \geq \Delta && [\mu] \quad \text{PK} \\
& \Delta(s') \geq 0,\ \forall s' && [\beta\pi(s')\lambda(s')] \quad \text{PART A} \\
& Q(\Delta(s'), s') \geq 0,\ \forall s' && [\beta\pi(s')\theta(s')] \quad \text{PART B} \\
& c \in [0,1]
\end{aligned}
$$

Thomas-Worrall show that $\Delta(s) \in [0, \bar{\Delta}(s)], \forall s$; $Q(\Delta, s)$ is decreasing and strictly concave in Δ; and **continuously differentiable** in $[0, \bar{\Delta}(s)]$.

The bounds $\bar{\Delta}(s)$ are such that

$$Q(\bar{\Delta}(s), s) = 0$$

Note that since Q is strictly decreasing, for any $\Delta(s) > \bar{\Delta}(s)$,

$$Q(\Delta(s), s) < 0$$

Thus, for every s', the upper bound of $\bar{\Delta}(s)$ covers all feasible allocations.

2 Characterizing Optimal Contract

Concavity of Q implies that the problem $(\mathbf{P})$ is convex. Taking the FOCs:

$$\begin{aligned} [c] : -u'(1-c) + \mu u'(c) &= 0 \\ [\Delta(s')] : \beta\pi(s')Q'(\Delta(s'), s') + \mu\beta\pi(s') & \\ +\beta\pi(s')\lambda(s') + \beta\pi(s')\theta(s')Q'(\Delta(s'), s') &= 0 \end{aligned}$$

Envelope condition:

$$Q'(\Delta, s) = -\mu$$

Substituting in the FOCs, we get

$$\begin{aligned} Q'(\Delta, s) &= -\frac{u'(1-c)}{u'(c)} \\ Q'(\Delta, s) &= (1+\theta(s))Q'(\Delta(s'), s') + \lambda(s) \end{aligned}$$

Strict concavity of Q implies that consumption is increasing in the promised value Δ_0. Given that Q' is continuous, c is a continuous function of Δ_0.

Given that c is increasing in Δ_0 and $\Delta_0 \in [0, \bar{\Delta}(s)]$, define

$$\begin{aligned} Q'(0, s) &= -\frac{u'(1-\underline{c}(s))}{u'(\underline{c}(s))} \\ Q'(\bar{\Delta}(s'), s') &= -\frac{u'(1-\bar{c}(s))}{u'(\bar{c}(s))} \end{aligned}$$

So

$$c(\Delta, s) \in [\underline{c}(s), \bar{c}(s)]$$

Claim 2.1. *There exists a unique $\underline{c}(s)$ such that*

$$Q'(0, s) = -\frac{u'(1-\underline{c}(s))}{u'(\underline{c}(s))}$$

Proof. Not that since Q is strictly decreasing, $Q'(0,s) < 0$. Since u satisfies the Inada conditions,

$$\lim_{c \to 1} -\frac{u'(1-\underline{c}(s))}{u'(\underline{c}(s))} = -\infty$$
$$\lim_{c \to 0} -\frac{u'(1-\underline{c}(s))}{u'(\underline{c}(s))} = 0$$

Also note that the ratio $-\frac{u'(1-\underline{C}(s))}{u'(\underline{C}(s))}$ is strictly decreasing. This gives us the result. □

Let g be such that

$$g(q) = -\frac{u'(1-q)}{u'(q)}$$

where g is decreasing. Note that

$$Q'(\Delta, s) = g(c(\Delta, s))$$

1. $\lambda(s') = \theta(s') = 0$. Then

$$\begin{aligned} Q'(\Delta, s) &= Q'(\Delta(s'), s') \\ -\frac{u'(1-c)}{u'(c)} &= -\frac{u'(1-c(s'))}{u'(c(s'))} \end{aligned}$$

   which implies that $c(s)$ is independent of s. For short hand, $c(s')$ really means $c(\Delta(s'), s')$. Moreover, $c(s') = c, \forall s$.

2. $\lambda(s') > 0$ and $\theta(s') = 0$. Then

$$\begin{aligned} Q'(\Delta, s) &= Q'(\Delta(s'), s') + \lambda(s') \\ g(c) &= g(c(s')) + \lambda(s') \end{aligned}$$

   which implies

$$\begin{aligned} g(c) &> g(c(s')) \\ c &< c(s') \end{aligned}$$

   Moreover, $\lambda(s') > 0 \Rightarrow \Delta(s') = 0 \Rightarrow c(s') = \underline{c}(s')$. The solution tomorrow is independent of $s \Rightarrow$"amnesia".

3. $\lambda(s') = 0$ and $\theta(s') > 0$. Then

$$\begin{aligned} g(c) &= (1+\theta(s'))g(c(s')) \\ g(c) &> g(c(s')) \\ c &> c(s') \end{aligned}$$

   and again, $\theta(s) > 0 \Rightarrow Q(\Delta(s), s) = 0 \Rightarrow \Delta(s) = \bar{\Delta}(s) \Rightarrow c(s) = \bar{c}(s)$. We continue having amnesia, where the solution tomorrow is indepedent of s_0.

4. $\lambda(s) > 0$ and $\theta(s) > 0$. Then

$$\begin{aligned} Q(\Delta(s), s) &= 0 \\ \Delta(s) &= 0 \end{aligned}$$

which implies that $\bar{\Delta}(s) = 0$

Proposition 2.2. *The optimal contract has the following form*

$$c(s) = \begin{cases} c & \text{if } c \in [\underline{c}(s), \bar{c}(s)] \\ \underline{c}(s) & \text{if } c < \underline{c}(s) \qquad \text{PART A binds} \\ \bar{c}(s) & \text{if } c > \bar{c}(s) \qquad \text{PART B binds} \end{cases}$$

Proposition 2.3. *Suppose* $y(s_1) > y(s_2)$, *then* $\bar{c}(s_1) > \bar{c}(s_2)$ *and* $\underline{c}(s_1) > \underline{c}(s_2)$

Proof. Consider

$$\begin{aligned} Q\left(\Delta + u(y(s_2)) - u(y(s_1)), s_1\right) = \max_{c,\Delta(s')} & \left\{ u(1-c) - u(1-y(s_1)) + \beta \sum_{s'} \pi(s') Q(\Delta(s'), s') \right\} \\ s.t. \qquad & u(c) - u(y(s_1)) + \beta \sum_{s'} \pi(s')\Delta(s') \geq \Delta + u(y(s_2)) - u(y(s_1)) \\ & \Delta(s') \geq 0, \ \forall s' \\ & Q(\Delta(s'), s') \geq 0, \ \forall s' \\ & c \in [0,1] \end{aligned}$$

$$\begin{aligned} = \max_{c,\Delta(s')} & \left\{ u(1-c) - u(1-y(s_2)) + \beta \sum_{s'} \pi(s) Q(\Delta(s'), s') \right\} + u(1-y(s_2)) - u(1-y(s_1)) \\ s.t. \qquad & u(c) - u(y(s_2)) + \beta \sum_{s'} \pi(s')\Delta(s') \geq \Delta \\ & \Delta(s') \geq 0, \ \forall s' \\ & Q(\Delta(s'), s') \geq 0, \ \forall s' \\ & c \in [0,1] \end{aligned}$$

Thus, for any Δ,

$$\begin{aligned} Q\left(\Delta + u(y(s_2)) - u(y(s_1)), s_1\right) &= Q(\Delta, s_2) + u(1-y(s_2)) - u(1-y(s_1)) \\ Q'\left(\Delta + u(y(s_2)) - u(y(s_1)), s_1\right) &= Q'(\Delta, s_2) \end{aligned}$$

$$
\begin{aligned}
Q\left(\bar{\Delta}(s_2) + u(y(s_2)) - u(y(s_1)), s_1\right) &= Q(\bar{\Delta}(s_2), s_2) + u(1 - y(s_2)) - u(1 - y(s_1)) \\
&= u(1 - y(s_2)) - u(1 - y(s_1)) \\
&> 0
\end{aligned}
$$

which implies that

$$\bar{\Delta}(s_2) + u(y(s_2)) - u(y(s_1)) < \bar{\Delta}(s_1)$$

Since Q is strictly concave,

$$
\begin{aligned}
Q'\left(\bar{\Delta}(s_2) + u(y(s_2)) - u(y(s_1)), s_1\right) &> Q'\left(\bar{\Delta}(s_1), s_1\right) \\
Q'(\bar{\Delta}(s_2), s_2) &> Q'\left(\bar{\Delta}(s_1), s_1\right) \\
g(\bar{c}(s_2)) &> g(\bar{c}(s_1)) \\
\bar{c}(s_2) &< \bar{c}(s_1)
\end{aligned}
$$

Similarly, one can show that $\underline{c}(s_1) > \underline{c}(s_2)$. □

Proposition 2.4. *$y(s) \in [\bar{c}(s), \underline{c}(s)]$ and $y(s_{min}) = \underline{c}(s_{min})$ and $y(s_{max}) = \bar{c}(s_{max})$*

Proof. We will only show half of the argument, and the other half is symmetric. First, we want to show that $y(s) \leq \bar{c}(s)$. This comes through the observation

$$Q(\bar{\Delta}(s), s) = u(1 - \bar{c}(s)) - u(1 - y(s)) + \beta \sum_{s'} \pi(s') Q(\Delta(s'), s') \tag{1}$$

$$= 0 \tag{2}$$

Since $Q(\Delta(s'), s') \geq 0$, this implies that $u(1 - \bar{c}(s)) - u(1 - y(s)) \leq 0$. Therefore, $y(s) \leq \bar{c}(s)$.

Next we want to show that $y(s_{max}) = \bar{c}(s_{max})$. Note that in the previous previous proof, we showed that if $y(s_1) > y(s_2)$, then

$$Q'(\bar{\Delta}(s_2), s_2) > Q'\left(\bar{\Delta}(s_1), s_1\right)$$

Thus, for all $s' \neq s_{max}$,

$$Q'(\bar{\Delta}(s'), s') > Q'\left(\bar{\Delta}(s_{max}), s_{max}\right)$$

This implies that

$$\bar{c}(s') < c(s_{max})$$

As we showed before, this implies that the participation constraint will bind in states $s' \neq s_{max}$ for agent B. In $s' = s_{max}$, we will have $c(s'_{max}) = c(s_{max}) = \bar{c}_{max}$, since consumption is within the bounds. This then implies that

$$\sum_{s'} \pi(s')Q(\Delta(s'), s') = 0$$
$$u(1 - \bar{c}_{max}) - u(1 - \bar{y}_{max}) = 0$$
$$y(s_{max}) = \bar{c}_{max}$$

□

3 Risk Sharing

Proposition 3.1. *Suppose that $\underline{c}(s_1) = \bar{c}(s_1) = y(s_1)$, then $\underline{c}(s) = \bar{c}(s) = y(s), \forall s$.*

Proof. This follows from a few observations. Suppose $\underline{c}(s_1) = \bar{c}(s_1) = y(s_1)$. In this case, the only feasible realizations of $\Delta(s_1)$ are 0, since this implies that $\bar{\Delta}(s_1) = 0$. Given this realization, we have the binding promise keeping constraint which implies

$$0 = u(y(s_1)) - u(y(s_1)) + \beta \sum_{s'} \pi(s')\Delta(s')$$

Since $\Delta(s') \geq 0$, this implies $\Delta(s') = 0 \; \forall \; s'$. Also observe

$$Q(\bar{\Delta}(s_1), s_1) = u(1 - y(s_1)) - u(1 - y(s_1)) + \beta \sum_{s'} \pi(s')Q(\Delta(s'), s')$$
$$= 0$$

Since $Q(\Delta(s'), s') \geq 0$, this implies that $Q(\Delta(s'), s') = 0$ for all s'. Together with this first observation, we have that $\bar{\Delta}(s') = 0$ for all s'. □

Proposition 3.2. *If $\bar{c}(y(s_{min})) < \underline{c}(y(s_{max}))$, then no first-best efficient allocation is sustainable.*