Economics 8107
Macroeconomic Theory
Recitation 3

Conor Ryan

February 6, 2017

Adapted From Manuel Amador's Notes

1 CARA Utility and Aggregate Shocks

Today we are going to talk about an environment with constant absolute risk aversion and aggregate shocks. It is an endowment economy, so we only have to worry about the consumer problems and the market clearing conditions. The consumer problem is

$$\begin{aligned} V(x^i, s) &= \max_a -\frac{1}{\gamma} e^{-\gamma(x^i - a)} + \beta E[V(x', s')|s] \\ \text{s.t.} \quad & x' = Ra + y^i(s') \end{aligned}$$

The endowment process $y^i(s)$ is characterized by an idiosyncratic and aggregate shock.

$$\begin{aligned} y^i(s) &= w^i(s) + y(s) \\ y(s') &= \phi_1 y(s) + (1 - \phi_1)\bar{y} + \epsilon(s') \\ w^i(s') &= \phi_2 w^i(s) + \eta^i(s') \\ \eta^i(s) &\sim N(0, \sigma_\eta^2) \\ \epsilon(s) &\sim N(0, \sigma_\epsilon^2) \end{aligned}$$

Note that the innovation shocks are iid. The market clearing conditions are

$$\int c(x^i, s)di = \int y^i(s)di$$
$$\int a(x^i, s)di = 0$$
$$\int x^i di = \int y^i(s)di$$

We will assume that R is constant and solve the consumer's problem as in class using guess and verify. As always, we have to make sure we pick the right guess. The typical guess for CARA utility is a value function that is log-linear in total cash in hand. However, in this environment we have persistent shocks, and we have to take into account that the agent can use the current state to predict future income. Therefore, our guess will be

$$V(x, s) = -\frac{R}{\gamma(R-1)} e^{-\gamma\left[Ax+By(s)+Dw^i(s)+F\right]}$$

Note that we still define $x = Ra + y(s) + w^i(s)$. So why do we include the endowment process terms twice in the value function? The terms outside of x reflect the information conveyed to the agent about future income. If the processes were i.i.d, these terms would be 0. From the envelope condition, we find that this implies a linear rule for consumption and, in turn, asset savings.

$$V_x(x, s) = u'(c(x, s))$$
$$A\frac{R}{(R-1)} e^{-\gamma\left[Ax+By(s)+Dw^i(s)+F\right]} = e^{-\gamma c(x,s)}$$
$$e^{-\gamma\left[Ax+By(s)+Dw^i(s)+F\right]+\log A\frac{R}{(R-1)}} = e^{-\gamma c(x,s)}$$
$$Ax + By(s) + Dw^i(s) + F - \frac{1}{\gamma}\log A\frac{R}{(R-1)} = c(x, s)$$

Since $c = x - a$, this implies

$$a(x, s) = (1 - A)x - By(s) - Dw^i(s) - F + \frac{1}{\gamma}\log A\frac{R}{(R-1)}$$

If we plug all of these equations back into the function equation, we find that $A = \frac{R-1}{R}$.

$$V(x^i, s) = -\frac{1}{\gamma}e^{-\gamma(c(x,s))} + \beta E[V(Ra(x, s) + y^i(s'), s')|s]$$

$$-\frac{R}{\gamma(R-1)}e^{-\gamma\left[Ax+By(s)+Dw^i(s)+F\right]} = -\frac{1}{\gamma}e^{-\gamma(Ax+By(s)+Dw^i(s)+F-\frac{1}{\gamma}\log A\frac{R}{(R-1)})}$$
$$+\beta E[-\frac{R}{\gamma(R-1)}e^{-\gamma\left[A(R((1-A)x-By(s)-Dw^i(s)-F+\frac{1}{\gamma}\log A\frac{R}{(R-1)})+y^i(s'))+By(s')+Dw^i(s')+F\right]}|s]$$

$$-\frac{R}{\gamma(R-1)}e^{-\gamma\left[Ax+By(s)+Dw^i(s)+F\right]} = -\frac{AR}{\gamma(R-1)}e^{-\gamma(Ax+By(s)+Dw^i(s)+F)}$$
$$-\beta\frac{R}{\gamma(R-1)}e^{-\gamma A(1-A)Rx}E[e^{-\gamma\left[A(-RBy(s)-RDw^i(s)-RF+\frac{1}{\gamma}\log A\frac{R}{(R-1)}+y^i(s'))+By(s')+Dw^i(s')+F\right]}|s]$$

Since this must hold for every x, we can collect the x terms and find that

$$\begin{aligned}
A(1-A)R - A &= 0\\
(1-A)R &= 1\\
1-A &= \frac{1}{R}\\
A &= 1-\frac{1}{R}\\
A &= \frac{R-1}{R}
\end{aligned}$$

Note that, in this case, $\log A\frac{R}{(R-1)} = 0$. Thus, our policy functions are

$$\begin{aligned}
a(x,s) &= (1-\frac{R-1}{R})x - By(s) - Dw^i(s) - F\\
c(x,s) &= \frac{R-1}{R}x + By(s) + Dw^i(s) + F
\end{aligned}$$

We can use the policy function, and or solution for A to better characterize next periods cash.

$$\begin{aligned}
x' &= Ra(x,s) + y^i(s')\\
x' &= R[x - c(x,s)] + y^i(s)\\
x' &= x(R - R\frac{R-1}{R}) - RBy(s) - RDw^i(s) - RF + y^i(s')\\
x' &= x - RBy(s) - RDw^i(s) - RF + y^i(s')
\end{aligned}$$

Then, we have

$$V(x',s') = -\frac{R}{\gamma(R-1)}\exp\{-\gamma\Big[\frac{R-1}{R}\big(x - RBy(s) - RDw^i(s) - RF + y^i(s')\big) + By(s') + Dw^i(s') + F\Big]\}$$

$$V_x(x',s') = \exp\{-\gamma\Big[\frac{R-1}{R}\big(x - RBy(s) - RDw^i(s) - RF + y^i(s')\big) + By(s') + Dw^i(s') + F\Big]\}$$

From the Euler Equation, we can solve for the remaining parameters.

$$u'(c(x,s)) = \beta RE[V_x(x',s')|s]$$

$$\exp\{-\gamma\big(\frac{R-1}{R}x + By(s) + Dw^i(s) + F\big)\} =$$

$$e^{\log\beta R}E[\exp\{-\gamma\Big[\frac{R-1}{R}\big(x - RBy(s) - RDw^i(s) - RF + y^i(s')\big) + By(s') + Dw^i(s') + F\Big]\}|s]$$

We can simplify terms, since the x and F appear on both sides, and are not affected by s',

$$\exp\{-\gamma\big(By(s) + Dw^i(s)\big)\} =$$

$$e^{\log\beta R}E[\exp\{-\gamma\Big[(1-R)\big(By(s) + Dw^i(s) + F\big) + \frac{R-1}{R}y^i(s') + By(s') + Dw^i(s')\Big]\}|s]$$

$$\exp\{-\gamma\big(RBy(s) + RDw^i(s) + (R-1)F\big) - \log\beta R\} =$$

$$E[\exp\{-\gamma\Big[\frac{R-1}{R}(y(s') + w^i(s')) + By(s') + Dw^i(s')\Big]\}|s]$$

$$\exp\{-\gamma\big(RBy(s) + RDw^i(s) + (R-1)F\big) - \log\beta R\} =$$

$$E[\exp\{-\gamma\Big[(\frac{R-1}{R} + B)y(s') + (\frac{R-1}{R} + D)w^i(s')\Big]\}|s]$$

$$\exp\{-\gamma\big(RBy(s) + RDw^i(s) + (R-1)F\big) - \log\beta R\} =$$

$$E[\exp\{-\gamma\Big[(\frac{R-1}{R} + B)(\phi_1 y(s) + (1-\phi_1)\bar{y} + \epsilon(s')) + (\frac{R-1}{R} + D)(\phi_2 w^i(s) + \eta^i(s))\Big]\}|s]$$

$$\exp\{-\gamma\big(RBy(s) + RDw^i(s) + (R-1)F\big) - \log\beta R\} =$$

$$\exp\{-\gamma\Big[(\frac{R-1}{R} + B)(\phi_1 y(s) + (1-\phi_1)\bar{y}) + (\frac{R-1}{R} + D)(\phi_2 w^i(s))\Big]\}$$

$$E[\exp\{-\gamma\Big[(\frac{R-1}{R} + B)\epsilon(s') + (\frac{R-1}{R} + D)\eta^i(s')\Big]\}|s]$$

$$\exp\{-\gamma\big(RBy(s) + RDw^i(s) + (R-1)F\big) - \log\beta R\} =$$

$$\exp\{-\gamma\Big[(\frac{R-1}{R} + B)(\phi_1 y(s) + (1-\phi_1)\bar{y}) + (\frac{R-1}{R} + D)(\phi_2 w^i(s))\Big]$$

$$+\big[\gamma(\frac{R-1}{R} + B)\big]^2\frac{\sigma_\epsilon^2}{2} + \big[\gamma(\frac{R-1}{R} + D)\big]^2\frac{\sigma_\eta^2}{2}\}$$

Not the $y(s)$ and $w^i(s)$ terms on each side of the equation. Since these must be equal for all s, it follows that

$$RB = (\frac{R-1}{R} + B)\phi_1$$

$$B = \frac{R-1}{R-\phi_1}\frac{\phi_1}{R}$$

$$D = \frac{R-1}{R-\phi_2}\frac{\phi_2}{R}$$

We should take a moment to see how these parameters fit with our expectations. Suppose the shocks are iid, then there is no persistent component and $\phi_1 = \phi_2 = 0$. In this case, both of the shocks drop out. Suppose the shocks are instead a random walk, then $\phi_1 = \phi_2 = 1$. In this case, B and D are both equal to $\frac{1}{R}$, which reflects that the expected future income from this components is simply equal to the income the agent received from each shock today.

The remaining terms will be grouped into F.

$$-\gamma(R-1)F = \log\beta R - \gamma\big[(\frac{R-1}{R} + B)(1-\phi_1)\bar{y}\big] + \big[\gamma(\frac{R-1}{R} + B)\big]^2\frac{\sigma_\epsilon^2}{2} + \big[\gamma(\frac{R-1}{R} + D)\big]^2\frac{\sigma_\eta^2}{2}$$

$$(R-1)F = -\frac{1}{\gamma}\log\beta R + \big[(\frac{R-1}{R-\phi_1})(1-\phi_1)\bar{y}\big] - \frac{1}{\gamma}\big[\gamma(\frac{R-1}{R-\phi_1})\big]^2\frac{\sigma_\epsilon^2}{2} - \frac{1}{\gamma}\big[\gamma(\frac{R-1}{R-\phi_2})\big]^2\frac{\sigma_\eta^2}{2}$$

$$F = -\frac{\log\beta R}{\gamma(R-1)} + \frac{1-\phi_1}{R-\phi_1}\bar{y} - \gamma\frac{R-1}{(R-\phi_1)^2}\frac{\sigma_\epsilon^2}{2} - \gamma\frac{R-1}{(R-\phi_2)^2}\frac{\sigma_\eta^2}{2}$$

We can now fully specify the consumption function as

$$c(x,s) = \frac{R-1}{R}x + \frac{R-1}{R-\phi_1}\frac{\phi_1}{R}y(s) + \frac{R-1}{R-\phi_1}\frac{\phi_2}{R}w^i(s) - \frac{\log\beta R}{\gamma(R-1)}$$

$$+\frac{1-\phi_1}{R-\phi_1}\bar{y} - \gamma\frac{R-1}{(R-\phi_1)^2}\frac{\sigma_\epsilon^2}{2} - \gamma\frac{R-1}{(R-\phi_2)^2}\frac{\sigma_\eta^2}{2}$$

Can we characterize something about equilibrium? Let's assume that the initial idiosyncratic shock is 0 for every agent, i.e. $w_{-1}^i = 0$. This is operationally equivalent to assuming that the initial shocks are mean 0. We could find a similar result if we looked for the limiting distribution of the AR processes. As such, if the initial shock is 0, the idiosyncratic shocks will be mean 0 for every period following.

$$\int w_0^i = \phi_2 \int w_{-1}^i di + \int \eta_0^i di = 0$$
$$\int w_t^i = \phi_2 \int w_{t-1}^i di + \int \eta_t^i di = 0$$

Then, the mean of total endowment process across all individuals is aggregate endowment level, the shock common to all agents.

$$\int y^i(s) di = \int y(s) di + \int w^i(s) di = y(s)$$

We can use the market clearing conditions to try to characterize the equilibrium interest rate.

$$y(s) = \int y^i(s) di = \int c(x^i, s) di$$
$$y(s) = \frac{R-1}{R} \int x^i di + \frac{R-1}{R-\phi_1}\frac{\phi_1}{R} y(s) + \frac{R-1}{R-\phi_1}\frac{\phi_2}{R}\int w^i(s) di - \frac{\log \beta R}{\gamma(R-1)}$$
$$+ \frac{1-\phi_1}{R-\phi_1}\bar{y} - \gamma\frac{R-1}{(R-\phi_1)^2}\frac{\sigma_\epsilon^2}{2} - \gamma\frac{R-1}{(R-\phi_2)^2}\frac{\sigma_\eta^2}{2}$$

$$y(s) = \frac{R-1}{R} y(s) + \frac{R-1}{R-\phi_1}\frac{\phi_1}{R} y(s) - \frac{\log \beta R}{\gamma(R-1)} + \frac{1-\phi_1}{R-\phi_1}\bar{y} - \gamma\frac{R-1}{(R-\phi_1)^2}\frac{\sigma_\epsilon^2}{2} - \gamma\frac{R-1}{(R-\phi_2)^2}\frac{\sigma_\eta^2}{2}$$

$$y(s)\big[1 - \frac{R-1}{R}(1 + \frac{\phi_1}{R-\phi_1})\big] = -\frac{\log \beta R}{\gamma(R-1)} + \frac{1-\phi_1}{R-\phi_1}\bar{y} - \gamma\frac{R-1}{(R-\phi_1)^2}\frac{\sigma_\epsilon^2}{2} - \gamma\frac{R-1}{(R-\phi_2)^2}\frac{\sigma_\eta^2}{2}$$

A slight tangent...

$$1 - \frac{R-1}{R}(1 + \frac{\phi_1}{R-\phi_1}) = \frac{1}{R}\big[R - R + 1 - (R-1)\frac{\phi_1}{R-\phi_1}\big]$$
$$= \frac{1}{R(R-\phi_1)}\big[R - \phi_1 - \phi_1 R + \phi_1\big]$$
$$= \frac{1}{R(R-\phi_1)}\big[R(1-\phi_1)\big]$$
$$= \frac{1-\phi_1}{R-\phi_1}$$

And we're back...

$$y(s)\frac{1-\phi_1}{R-\phi_1} = -\frac{\log\beta R}{\gamma(R-1)} + \frac{1-\phi_1}{R-\phi_1}\bar{y} - \gamma\frac{R-1}{(R-\phi_1)^2}\frac{\sigma_\epsilon^2}{2} - \gamma\frac{R-1}{(R-\phi_2)^2}\frac{\sigma_\eta^2}{2}$$

$$\frac{1-\phi_1}{R-\phi_1}(y(s)-\bar{y}) = -\frac{\log\beta R}{\gamma(R-1)} - \gamma\frac{R-1}{(R-\phi_1)^2}\frac{\sigma_\epsilon^2}{2} - \gamma\frac{R-1}{(R-\phi_2)^2}\frac{\sigma_\eta^2}{2}$$

Alas, there can be no interest rate that satisfies this equation for all states. The interest in this equation is going to depend on the level of the aggregate endowment today. If $\phi_1 = 1$, then the state-variant term drops out. In this case, the aggregate shock is a random walk. Then we have, similar to the problem in class,

$$\frac{\log\beta R}{\gamma(R-1)} = -\gamma\frac{R-1}{(R-\phi_1)^2}\frac{\sigma_\epsilon^2}{2} - \gamma\frac{R-1}{(R-\phi_2)^2}\frac{\sigma_\eta^2}{2}$$
$$\log\beta R = -\gamma\frac{\sigma_\epsilon^2}{2} - \gamma(\frac{R-1}{R-\phi_2})^2\frac{\sigma_\eta^2}{2}$$

The left-hand-side is strictly increasing in R, and the right hand side is strictly decreasing in R, which means we will have a unique solution R^* that clears our markets. Note that solving this equation is equivalent to solving that the constant F, which represents the drift, must be 0. We can see a similar result as we saw in class, than cash in hand will spread out. The consumption policy is

$$c(x,s) = \frac{R-1}{R}x + \frac{1}{R}y(s) + \frac{R-1}{R-\phi_1}\frac{\phi_2}{R}w^i(s)$$

Recall that the next period cash in hands can be expressed as

$$x'(x,s,s') = x - RBy(s) - RDw^i(s) - RF + y^i(s')$$
$$x'(x,s,s') = x - y(s) - R\frac{R-1}{R-\phi_2}\frac{\phi_2}{R}w^i(s) + y(s') + w^i(s')$$
$$x'(x,s,s') - y(s') = x - y(s) - \frac{\phi_2(R-1)}{R-\phi_2}w^i(s) + w^i(s')$$
$$x'(x,s,s') - y(s') = x - y(s) - \frac{\phi_2(R-1)}{R-\phi_2}w^i(s) + \phi_2 w^i(s) + \eta^i(s')$$
$$x'(x,s,s') - y(s') = x - y(s) + \frac{\phi_2 R - \phi_2^2 - \phi_2 R - \phi_2}{R-\phi_2}w^i(s) + \eta^i(s')$$
$$[x'(x,s,s') - y(s')] = [x - y(s)] + \frac{\phi_2(1-\phi_2)}{R-\phi_2}w^i(s) + \eta^i(s')$$

Thus, there is a random walk component to an agents asset position relative to the aggregate.

Why should we think that a constant interest rate is consistent with both the random walk case (full persistence), but not when aggregate shocks have imperfect persistence? Does it work if the aggregate shocks are i.i.d.? The key is that the interest rate in the economy reflects the agents belief about the future. An agent that believes she will be richer in the future wants to borrow. This pushes the interest rate up. An agent that believes she will be poorer in the future wants to save, which has the opposite effect. When shocks are not fully persistent, the aggregate state today may be above or below the expected state tomorrow. In the simple i.i.d. case, whenever the economy experiences a bad shock, the agents expect things to be better tomorrow. Through these mechanisms, these models predict high interest rates during bad times and low interest rates during good times.