Economics 8107
Macroeconomic Theory
Recitation 2

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Adapted From Manuel Amador's Notes

1 Balanced Growth Path

Consider the following deterministic growth model with leisure and capital investment

- Household's preferences:

$$\begin{aligned} &\max \sum_t \beta^t U(c_t, 1-n_t) \\ s.t. \quad & \sum_t p_t(c_t + x_t) \leq \sum_t p_t(w_t n_t + r_t k_t) \\ & k_{t+1} = x_t + (1-\delta)k_t \\ & c_t, x_t, k_{t+1} \geq 0 \\ & 0 \leq n_t \leq 1 \end{aligned}$$

- Production function: $y_t = F(k_t, \lambda^t n_t)$, which is homogenous of degree 1
- Feasibility:

$$c_t + k_{t+1} = F(k_t, \lambda^t n_t) + (1-\delta)k_t$$

Definition 1.1. A Balanced Growth Path is given by an initial capital k_0 and λ such that it is optimal to set

$$\begin{aligned} c_t &= c_0 \lambda^t \\ k_t &= k_0 \lambda^t \\ n_t &= n_0 \\ w_t &= \lambda^t w_0 \\ r_t &= r_0 \end{aligned}$$

Note that a balanced growth path is characterized by consumption, capital, and investment all grow at a constant rate and the interest rate is constant. However, hours worked obviously cannot grow exponentially, but wages do grow.
Question. *Why do we care about Balanced Growth Path?*

Consider the following **Kaldor's Growth Facts (1961):**

1. Output per worker grows over time at relatively constant and positive rate.
2. Capital per worker grows over time at relatively constant and positive rate.
3. Output per worker and capital per worker grow at similar rates, so K_t/Y_t is relatively constant over time.
4. The real return to capital $r_t - \delta$ is relatively constant over time.
5. The share of labor $\frac{w_t l_t}{y_t}$ and share of capital $\frac{r_t l_t}{y_t}$ are relatively constant.
6. Growth rate of ouput per worker differs across countries.

Therefore, if we want to write a macro model, we would like that in its steady states, it establishes Kaldor's facts.
Claim 1.2. *With the BGP requirements, the equilibrium in our model will satisfy Kaldo's growth facts.*

Proof. Homework □

Utility Function Consistent with BGP

Start with the Euler equation of our model:

$$U_c(c_t, 1-n_t) = \beta(r_t + 1 - \delta)U_c(c_{t+1}, 1-n_{t+1})$$

Imposing BGP, we have that $\forall t$,

$$U_c(c_0\lambda^t, 1-n_0) = \beta(r_0 + 1 - \delta)U_c(c_0\lambda^{t+1}, 1-n_0) \tag{1}$$

Note that

$$\lambda^t = e^{t \log \lambda} \Rightarrow \frac{d\lambda^t}{dt} = \lambda^t \log \lambda$$

Taking derivatives of both sides of (1) with respect to t, we have

$$U_{cc}(c_0\lambda^t, 1-n_0)\lambda^t \log \lambda c_0 = \beta(r_0 + 1 - \delta)U_{cc}(c_0\lambda^{t+1}, 1-n_0)\lambda^{t+1} \log \lambda c_0 \tag{2}$$

Dividing (2) by (1), we have that $\forall t$,

$$\frac{U_{cc}(c_t, 1-n_0)}{U_c(c_t, 1-n_0)}c_t = \frac{U_{cc}(c_{t+1}, 1-n_0)}{U_c(c_{t+1}, 1-n_0)}c_{t+1}$$

Hence, the ratio $\frac{U_{cc}}{U_c}c$ must be independent of consumption. In particular, we can write that

$$\frac{U_{cc}(c, 1-n)}{U_c(c, 1-n)}c = -\gamma(1-n)$$

Note: $\gamma(1-n)$ is a constant that could still, in principle, depend on n.

Exercise: Show that the solution to the above differential equation is of the form

$$U(c, 1-n) = a + \frac{c^{1-\gamma}}{1-\gamma}v(1-n)$$

for a function $v(\cdot)$ that only depends on $1-n$ and some constants a, γ where $\gamma \neq 1$. As of yet, both constants a and *gamma* may depend on n.

The properties of $v(\cdot)$ will be determined by the requirements that utility is increasing and strictly concave in both arguments.

What about the intratemporal condition? The first order condition for consumption and labor give us

$$\frac{U_{1-n}(c_t, 1-n_t)}{U_c(c_t, 1-n_t)} = w_t$$
$$\frac{U_{1-n}(c_t, 1-n_t)}{U_c(c_t, 1-n_t)} = \lambda^t w_0$$

Given the formulation that we have so far,

$$U_c = (1-\gamma)c^{-\gamma}\frac{v(1-n)}{1-\gamma}$$
$$U_c = (1-\gamma)c^{-\gamma}\tilde{v}(1-n)$$

$$U_{1-n} = a' + c^{1-\gamma}\tilde{v}(1-n)(-\gamma' \ln c) + c^{1-\gamma}\tilde{v}'(1-n)$$

Then, we have

$$\frac{a' + c^{1-\gamma}(-\tilde{v}(1-n)\gamma' \ln c + \tilde{v}'(1-n))}{(1-\gamma)c^{-\gamma}\tilde{v}(1-n)} = \lambda^t w_0$$
$$\frac{a' + (\lambda^t c_0)^{1-\gamma}(-\tilde{v}(1-n)\gamma' \ln(\lambda^t c_0) + \tilde{v}'(1-n))}{(-\gamma)(\lambda^t c_0)^{1-\gamma}\tilde{v}(1-n)} = \lambda^t w_0$$

We need this equation to hold for all t. So intuitively, we must have the LHS increasing multiplicatively with λ^t, just like the RHS. It turns out that the necessary and sufficient conditions for this are that a' and γ' are 0, i.e., the constants do not vary with the labor supply.

Interpretation

In BGP, my preferences have to be such that percentage change in MRS will entail some fixed percentage between consumption today and tomorrow. In other words, intertemporal elasticity of substitution between consumption today and tomorrow has to be constant. Define $V(c_t, c_{t+1}) = u(c_t) + \beta u(c_{t+1})$. Then by definition of elasticity of substitution

$$\frac{d(V_{c_t}/V_{c_{t+1}})}{d(c_t/c_{t+1})}\frac{c_t/c_{t+1}}{V_{c_t}/V_{c_{t+1}}} = -\gamma$$

Using the fact that $V(c_t, c_{t+1}) = u(c_t) + \beta u(c_{t+1})$ and $c_{t+1} = \lambda c_t$, we obtain that

$$\frac{u"(c_t)}{u'(c_t)}c_t = -\gamma$$

2 An application: Ricardian equivalence

Here we study the impact of government tax and debt policy on aggregate economic outcomes. Specifically, we consider a government who needs to finance some exogenous stream of expenditures in an economy with complete markets. The government can levy lump-sum taxes and issue debt. We establish the Ricardian equivalence: we show that the manner in which the stream of expenditures is financed does not matter for aggregate outcomes.

2.1 The economic environment

To see this point, consider the same economy as before but add a government with some exogenous per-capita expenditure plan $g = \{g_t(s^t) : t \geq 0, s^t \in S^t\}$. For simplicity, we assume that government expenditures are thrown away. The government chooses the size of per-capita lump-sum taxes, $\tau = \{\tau_t(s^t) : t \geq 0, s^t \in S^t\}$, and of (state contingent) per-capita debt issuance, $B \equiv \{B_t(s^t) : t \geq 0, s^t \in S^t\}$, subject to the sequential budget constraint:

$$g_t\left(s^t\right) + B_t\left(s^t\right) = \sum_{s_{t+1}\in S} Q_{t+1}\left(s_{t+1}|s^t\right) B_{t+1}\left(s^t, s_{t+1}\right) + \tau_t\left(s^t\right) \tag{3}$$

for all times $t \geq 0$ and every history $s^t \in S^t$, where $q_{0t}(s^t) = Q_t(s_t \mid s^{t-1}) \times Q_{t-1}(s_{t-1} \mid s^{t-2}) \ldots \times Q_1(s_1 \mid s_0)$. For simplicity, we assume that, at time $t = 0$, the government starts with no debt, $B_0(s_0) = 0$.

Agent $i \in \{1, \ldots, I\}$ maximizes

$$\sum_{t\geq 0}\sum_{s^t\in S^t} \pi_{0t}(s^t)u_i\left[c_{it}(s^t)\right],$$

with respect to a consumption and asset holdings plan, $(c_i, a_i) = \{c_{it}(s^t), a_{it}(s^t) : t \geq 0, s^t \in S^t\}$, subject to subject to

$$c_{it}\left(s^t\right) + \sum_{s_{t+1}\in S} Q_{t+1}\left(s_{t+1}|s^t\right) a_{i,t+1}\left(s^t, s_{t+1}\right) + \tau_t\left(s^t\right) = y_{it}\left(s^t\right) + a_{it}\left(s^t\right) \tag{4}$$

$$a_{i,t+1}(s^t, s_{t+1}) \geq \overline{A}_i(s^{t+1}) \tag{5}$$

for all times $t \geq 0$ and histories $s^t \in S^t$, and $a_{i0}(s_0) = 0$.

The borrowing limits $\overline{A}_i(s^t)$ must satisfy

$$\overline{A}_i\left(s^t\right) = y^i\left(s^t\right) + \sum_{s^{t+1}|s^t} Q\left(s^{t+1}|s^t\right) \overline{A}_i\left(s^{t+1}\right) \tag{6}$$

We define an equilibrium as follows:
Definition 2.1. Given a stream of government expenditures, g, a competitive equilibrium consists of a debt and tax policy, $\{\tau, B\}$, a consumption and asset allocation for each agentsm $\{c_i, a_i\}_{i\in I}$, a price system Q, and borrowing limits, $\{\overline{A}_i\}_{i\in I}$ such that

- $\{B, \tau\}$ satisfies the government sequential budget constraint (3) given g and Q.
- For all $i \in \{1, \ldots, I\}$, $\{c_i, a_i\}$ solves agent i's problem given Q and $\overline{A}_i$;
- The borrowing limits $\{\overline{A}_i\}$ satisfy the natural borrowing limit equations;
- And markets clear

$$\sum_{i=1}^{I}\left[c_{it}(s^t) + g_t(s^t)\right] = \sum_{i=1}^{I} y_{it}(s^t)$$

$$\sum_{i=1}^{I} a_{it}(s^t) = I \times B_t(s^t).$$

2.2 Sequential vs. time zero budget sets

In what follows we assume that $|B_t(s^t)| \leq \bar{B}$ for some sufficiently large and finite $\bar{B}$.

For the government, the time-zero inter-temporal budget constraint equates the present value of per capita expenditures to the present value of per capita lump sum taxes:

$$\sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t) g_t(s^t) = \sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)\tau_t(s^t). \tag{7}$$

For an agent, the time-zero inter-temporal budget constraint is:

$$\sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t)\left[c_t(s^t) + \tau_t(s^t)\right] = \sum_{t\geq 0}\sum_{s^t\in S^t} q_{0t}(s^t) y_t(s^t). \tag{8}$$

It is the same constraint as before, except for the fact that the agents' expenditure include the payment of a lump sum tax, $\tau_t(s^t)$.

Proposition 2.2. *Let $q_{0t}(s^t) \equiv Q_t(s_t \,|\, s^{t-1}) \times Q_{t-1}(s_{t-1} \,|\, s^{t-2}) \times \ldots \times Q_1(s_1 \,|\, s_0)$. Then, the consumption and asset holding plan $\{c_i, a_i\}$ is budget feasible for agent i in sequential markets given Q if and only if it is budget feasible in time-zero markets given q and*

$$a_{it}(s^t) = \sum_{k \geq 0} \sum_{s^{t+k} \succeq s^t} \frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)} \left[c_{t+k}(s^{t+k}) + \tau_{t+k}(s^{t+k}) - y_{t+k}(s^{t+k})\right], \tag{9}$$

for all times $t \geq 0$ and histories $s^t \in S^t$.

Proposition 2.3. *Let $q_{0t}(s^t) \equiv Q_t(s_t \,|\, s^{t-1}) \times Q_{t-1}(s_{t-1} \,|\, s^{t-2}) \times \ldots \times Q_1(s_1 \,|\, s_0)$. Then, a policy $\{g, \tau, B\}$ is budget feasible for the government in sequential markets given Q if and only if it budget feasible in time-zero markets given q and*

$$B_t(s^t) = \sum_{k \geq 0} \sum_{s^{t+k} \succeq s^t} \frac{q_{0t+k}(s^{t+k})}{q_{0t}(s^t)} \left[\tau_{t+k}(s^{t+k}) - g_{t+k}(s^{t+k})\right], \tag{10}$$

for all times $t \geq 0$ and histories $s^t \in S^t$.

2.3 The main result

The main proposition for this section is that equilibria do not depend on the timing of taxes and debt. To see how this result obtains, follow the logic of Corollary 2.2 and 2.3. From 2.2 , we know that the agent chooses her consumption plan "as if" she were facing the single time-zero inter-temporal budget constraint (8). Notice in particular that the agent only cares about the present value of lump-sum taxes she will pay to the government – the precise timing of taxes does not matter.

Now, look at the the government inter-temporal budget constraint (2.3): it implies that the present value of lump sum taxes must be equal to the present value of expenditures. Put differently, if we substitute the government time zero inter-temporal budget constraint in the agent time zero inter-temporal budget constraint, we obtain:

$$\sum_{t \geq 0} \sum_{s^t \in S^t} q_{0t}(s^t) \left[c_t(s^t) + g_t(s^t)\right] = \sum_{t \geq 0} \sum_{s^t \in S^t} q_{0t}(s^t) y_t(s^t). \tag{11}$$

Hence, government policy only constrains agents' consumption choice via the present value of its per capita expenditure. The details of public finance are irrelevant. This means that if the government changes its stream of taxes to τ', or if it changes its debt policy to B', but keeps its expenditure the same, then the agent's original consumption plan remains optimal.

The asset holdings must change however, and are determined by (9). For example, if the government reduces $\tau_t(s^t)$ and increases $\tau_{t+k}(s^{t+k})$, then it must increase the amount of debt it issues, $B_{t+1}(s^t, s_{t+1})$. One sees from (9) that the asset holdings $a_{it+1}(s^t, s_{t+1})$ of the agent, $a_{it+1}(s^t, s_{t+1})$, must increase as well. Indeed, the agent saves more because he anticipates the future increase in lump sum taxes at time $t + k$. Formally, we obtain:

Proposition 2.4. *Consider an equilibrium* $\{\tau, B, c_i, a_i, Q\}$ *and let*

$$q_{0t}(s^t) \equiv Q_t(s_t \,|\, s^{t-1}) \times Q_{t-1}(s_{t-1} \,|\, s^{t-2}) \times \ldots \times Q_1(s_1 \,|\, s_0).$$

Consider any stream of taxes $\hat{\tau}$ *such that:*

$$\sum_{t \geq 0} \sum_{s^t \in S^t} q_{0t}(s^t) \hat{\tau}_t(s^t) = \sum_{t \geq 0} \sum_{s^t \in S^t} q_{0t}(s^t) \tau_t(s^t). \tag{12}$$

Then $\{\hat{\tau}, \hat{B}, c_i, \hat{a}_i, Q\}$ *is an equilibrium, where* $\hat{B}$ *and* $\hat{a}$ *are given by* (9) *and* (10) *given* $\hat{\tau}$.