Economics 8107
Macroeconomic Theory
Recitation 1

Conor Ryan

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Overlapping Generation Model

An **overlapping generation model** assumes there are an infinite number of finitely lived consumers. In each period, a new generation is born, which is indexed by the date of birth. In the more coarse versions of this model, the consumers fit the "generation" notion. An individual agent lives for 2 or 3 periods, which may follow some kind of work pattern, i.e. early career, late career, and retirement. In more detailed model, there may be a "generation" for each birth year, leading to 50, 80, 100 generations in the model.

Today we will focus on the simplest form, a 2 generation model. Each consumer lives 2 periods—young and old age. By (e_t^t, e_{t+1}^t) we denote generation t's endowment of the consumption good in the first and second period of their life. The superscript reflects the birth period of the generation, and the subscript reflects the current period. Likewise, (c_t^t, c_{t+1}^t) denotes the consumption allocation of generation t. Hence in time t there are two generations alive, one old generation that was born in $t-1$ with endowment e_t^{t-1} and consumption c_t^{t-1} and one young generation born in t that has endowment e_t^t and consumption c_t^t . Moreover, there is an "initial old" generation. If the model starts in period 0, there is an old generation in $t = 0$ that was born outside of the model, so to speak. This generation has endowment e_0^{-1} and consumes c_0^{-1}.

$\frac{\text{period}}{\text{generation}}$	0	1	2	3	4	$\cdots$
-1	(c_0^{-1}, e_0^{-1})					
0	(c_0^0, e_0^0)	(c_1^0, e_1^0)				
1		(c_1^1, e_1^1)	(c_2^1, e_2^1)			
2			(c_2^2, e_2^2)	(c_3^2, e_3^2)		
3				(c_3^3, e_3^3)	(c_4^3, e_4^3)	
$\vdots$					$\cdots$	

1 Equilibrium

1.1 Definitions

In this section, we will revisit the definitions of Arrow-Debreu and sequential markets equilibria. Note that since each household only lives for two periods, utility and budget constraints are defined only over two periods. We will specify that preferences for a representative consumer in each generation t are given by $U_t(c_t^t, c_{t+1}^t)$. We also have to write a separate problem for the initial old generation.

Definition 1.1. An Arrow-Debreu Equilibrium is allocation $\left\{c_0^{-1}, \left(c_t^t, c_{t+1}^t\right)_{t=0}^{\infty}\right\}$ and price system $\{p_t\}$ such that

Given $\{p_t\}$, household born in period t chooses (c_t^t, c_{t+1}^t) that solves

$$\begin{aligned}
\max_{c_t^t, c_{t+1}^t} \quad & U_t(c_t^t, c_{t+1}^t) \\
s.t. \quad & p_t c_t^t + p_{t+1} c_{t+1}^t \leq p_t e_t^t + p_{t+1} e_{t+1}^t \\
& c_t^t, c_{t+1}^t \geq 0
\end{aligned}$$

Given p_0, the initial old consumer chooses c_0^{-1} that solves

$$\begin{aligned}
\max_{c_0^{-1}} \quad & U_{-1}(c_0^{-1}) \\
s.t. \quad & p_0 c_0^{-1} \leq p_0 e_0^{-1} \\
& c_0^{-1} \geq 0
\end{aligned}$$

Market Clearance: $\forall t$,

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t$$

Definition 1.2. A Sequential markets Equilibrium is allocation $\left\{c_0^{-1}, \left(c_t^t, c_{t+1}^t, b_{t+1}^t\right)_{t=0}^{\infty}\right\}$ and price system $\{r_t\}$ such that

Given $\{r_t\}$, $\left(c_t^t, c_{t+1}^t, b_{t+1}^t\right)$ solves

$$\begin{aligned}
\max_{c_t^t, c_{t+1}^t} \quad & U_t(c_t^t, c_{t+1}^t) \\
s.t. \quad & c_t^t + b_{t+1}^t \leq e_t^t \\
& c_{t+1}^t \leq e_{t+1}^t + (1 + r_{t+1}) b_{t+1}^t \\
& c_t^t, c_{t+1}^t \geq 0
\end{aligned}$$

The initial old consumer chooses c_1^0 that solves

$$\begin{aligned}
\max_{c_0^{-1}} \quad & U_{-1}(c_0^{-1}) \\
s.t. \quad & c_0^{-1} \leq e_0^{-1} \\
& c_0^{-1} \geq 0
\end{aligned}$$

Market Clearance: $\forall t$,

$$\begin{aligned}
c_t^{t-1} + c_t^t &= e_t^{t-1} + e_t^t \\
b_{t+1}^t &= 0
\end{aligned}$$

Notice that we do not need a No-Ponzi condition in this environment. In the infinitely-lived consumer, the borrowing limit presents the consumer from continuously rolling over her debt. Here, that is not necessary because the consumer only lives for two periods. This is included in the model explicitly by simply denying the old generation access to the credit market. We could write the problem in another way:

Definition 1.3 (Alternative SME)**.** A Sequential markets Equilibrium is allocation $\left\{c_0^{-1}, \left(c_t^t, c_{t+1}^t, b_{t+1}^t\right)_{t=0}^{\infty}\right\}$ and price system $\{r_t\}$ such that

Given $\{r_t\}$, $\left(c_t^t, c_{t+1}^t, b_{t+1}^t\right)$ solves

$$\begin{aligned}
\max_{c_t^t, c_{t+1}^t} \quad & U_t(c_t^t, c_{t+1}^t) \\
s.t. \quad & c_t^t + b_{t+1}^t \leq e_t^t \\
& c_{t+1}^t + b_{t+2}^t \leq e_{t+1}^t + (1 + r_{t+1}) b_{t+1}^t \\
& c_t^t, c_{t+1}^t \geq 0 \\
& b_{t+2}^t \geq 0
\end{aligned}$$

The initial old consumer chooses c_1^0 that solves

$$\begin{aligned}
\max_{c_0^{-1}} \quad & U_{-1}(c_0^{-1}) \\
s.t. \quad & c_0^{-1} + b_1^{-1} \leq e_0^{-1} \\
& c_0^{-1} \geq 0 \\
& b_1^{-1} \geq 0
\end{aligned}$$

Market Clearance: $\forall t$,

$$\begin{aligned} c_t^{t-1} + c_t^t &= e_t^{t-1} + e_t^t \\ b_{t+1}^{t-1} + b_{t+1}^t &= 0 \end{aligned}$$

Now the borrowing limit is written expressly into the problem. The old generation is not allowed to borrow, since they will be unable to pay back the debt. The old generation will then optimally choose not to save, since they would rather increase consumption. This has the implication that the old generation will never save nor borrow, so it is identical to a model where we do not give them the option to access the credit market.

Importantly, this means that the only savers/borrows are the young generation. And since there is a representative consumer for each generation, there will be no borrowing or saving.

2 Pareto Efficiency

Definition 2.1. An allocation $\left\{c_0^{-1}, \left(c_t^t, c_{t+1}^t\right)_{t=0}^{\infty}\right\}$ is feasible if $\forall t \geq 0, c_t^{t-1}, c_t^t \geq 0$ and $c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t$.

An allocation $\left\{c_0^{-1}, \left(c_t^t, c_{t+1}^t\right)_{t=0}^{\infty}\right\}$ is Pareto efficient if

1. It is feasible
2. There exists no other **feasible** allocation $\left\{\hat{c}_0^{-1}, \left(\hat{c}_t^t, \hat{c}_{t+1}^t\right)_{t=0}^{\infty}\right\}$ such that

$$\begin{aligned} \forall t \geq 0,\ U_t(\hat{c}_t^t, \hat{c}_{t+1}^t) &\geq U_t(c_t^t, c_{t+1}^t) \\ U_{-1}(\hat{c}_0^{-1}) &\geq U_{-1}(c_0^{-1}) \end{aligned}$$

   with one strict inequality for at least one $t \geq 0$

3 Inefficient Equilibria

Question. *What do you think is the First Welfare Theorem in a OLG model?*

Suppose $\forall t, U_t$ *is locally non-satiated and* $\forall t \geq 0,\ e_t^{t-1}, e_t^t \geq 0$*. Let* $\left\{c_0^{-1}, \left(c_t^t, c_{t+1}^t\right)_{t=0}^{\infty}\right\}$ *and* $\{p_t\}$ *be an Arrow-Debreu equilibrium, then* $\left\{c_0^{-1}, \left(c_t^t, c_{t+1}^t\right)_{t=0}^{\infty}\right\}$ *is Pareto efficient.*
Question. *Does the First Welfare Theorem hold?*

Unlike infinitely-lived consumer model, the First Welfare Theorem does not always hold in an OLG model. Consider the following environment where

$$\begin{aligned} U_t(c_t^t, c_{t+1}^t) &= \log c_t^t + \log c_{t+1}^t \\ (e_t^t, e_{t+1}^t) &= (4, 2) \end{aligned}$$

We have already shown that the generations will not save or borrow. Therefore, the agents will always simply consume their endowments. Thus, AD equilibrium is $\forall t \geq 0$,

$$\begin{aligned} c_t^{t-1} &= 2 \\ c_t^t &= 4 \end{aligned}$$

Consider another feasible allocation $\left\{\hat{c}_0^{-1}, \left(\hat{c}_t^t, \hat{c}_{t+1}^t\right)_{t=0}^{\infty}\right\}$ where $\forall t \geq 0$,

$$\begin{aligned} \hat{c}_t^{t-1} &= 3 \\ \hat{c}_t^t &= 3 \end{aligned}$$

Note that

$$\log 3 > \log 2$$
$$\log \hat{c}_0^{-1} > \log c_0^{-1}$$
$$\log 3 + \log 3 > \log 2 + \log 4$$
$$\log \hat{c}_t^t + \log \hat{c}_{t+1}^t > \log c_t^t + \log c_{t+1}^t$$

Thus, the ADE allocation $\left\{c_0^{-1}, \left(c_t^t, c_{t+1}^t\right)_{t=0}^{\infty}\right\}$ in this case is **not** Pareto efficient.

What does it look like to implement this Pareto improvement? The social planner would force a transfer from the richer (or working) young generation to the older generation. This is one argument for social retirement programs like social security.

4 A Diversion to the Research Frontier

If we create a more detailed model in the agents live longer than two periods, the economy will generate savings and borrowing, depending on what the endowment process looks like. If there is a final period, that agent will always have no borrowing (by constraint) and no savings (by optimality). This assumption can also be relaxed if there is "final period." Suppose instead agents face some probability of death every period δ_t.

Many OLG models move in this direction. One phenomenon that some economists are trying to explain are bequest patterns—the act of leaving money for your children. In the simple model we have described, you would conclude that bequests are driven entirely by the randomness of death. Suppose in our simple model, there is a deterministic endowment pattern y_t and a random probability of survival $1-\delta_t$. Then, we can write a dynamic problem for an agent that is t periods old as

$$V_t(A_t) = \max_{c_t} u(c_t) + \beta(1-\delta_t)V_{t+1}(A_{t+1})$$
$$c_t + qA_{t+1} = A_t + y_t$$

In this model, bequests are simply precautionary savings for an agent that met a random end. Alternatively, you might think that people value leaving money for their children. In that model, we could write

$$\begin{aligned} V_t(A_t) &= \max_{c_t} u(c_t) + \beta\big[\delta_t v(A_{t+1}) + (1-\delta_t)V_{t+1}(A_{t+1})\big] \\ c_t + qA_{t+1} &= A_t + y_t \end{aligned}$$

where v is some preference over leaving an inheritance. The job market paper that will presented here this Thursday explores a more detailed version of this sort of model to differentiate between these bequest motives.

5 Equivalence Results

Proposition 5.1. *(ADE⇒SME)*

Let $\left\{\hat{c}_1^0, \left(\hat{c}_t^t, \hat{c}_{t+1}^t\right)_{t=1}^{\infty}\right\}$ *and* $\{p_t\}$ *be an Arrow-Debreu equilibrium, then there exists a corresponding sequential markets equilibrium* $\left\{\tilde{c}_1^0, \left(\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{b}_{t+1}^t\right)_{t=1}^{\infty}\right\}$ *and* $\{r_t\}$ *such that* $\forall t \geq 1$,

$$\begin{aligned} \tilde{c}_t^{t-1} &= \hat{c}_t^{t-1} \\ \tilde{c}_t^t &= \hat{c}_t^t \\ \tilde{b}_{t+1}^t &= e_t^t - \hat{c}_t^t \\ 1 + r_{t+1} &= p_t/p_{t+1} \\ 1 + r_1 &= 1/p_1 \end{aligned}$$

Proposition 5.2. *(SME⇒ADE)*

Let $\left\{\tilde{c}_1^0, \left(\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{b}_{t+1}^t\right)_{t=1}^{\infty}\right\}$ *and* $\{r_t\}$ *be a sequential markets equilibrium, then there exists a corresponding Arrow-Debreu equilibrium* $\left\{\hat{c}_1^0, \left(\hat{c}_t^t, \hat{c}_{t+1}^t\right)_{t=1}^{\infty}\right\}$ *and* $\{p_t\}$ *such that* $\forall t \geq 1$,

$$\begin{aligned} \hat{c}_t^{t-1} &= \tilde{c}_t^{t-1} \\ \hat{c}_t^t &= \tilde{c}_t^t \\ p_1 &= \frac{1}{1 + r_1} \\ p_{t+1} &= \frac{p_t}{1 + r_{t+1}} \end{aligned}$$