Economics 8106
Macroeconomic Theory
Recitation 2

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Outline:

- Sequential Trading with Arrow Securities
- Lucas Tree Asset Pricing Model
- The Equity Premium Puzzle

1 Sequential Trading with Arrow Securities

In this first section, we will study the sequential trading formulation of an endowment economy and its implications on asset pricing. The reference text for this portion is sections 8.7 and 8.8 in Ljungqvist and Sargent.

First consider the most basic stochastic endowment economy. The household will solve the following problem,

$$\max_{(c_t(s^t), a_{t+1}(s^t))} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t|s_0) u(c_t(s^t))$$

$$s.t.$$

$$\sum_{t} \sum_{s^t} q_t(s^t) c_t(s^t) \leq \sum_{t} \sum_{s^t} q_t(s^t) y_t(s^t)$$

$$c_t \geq 0$$

$$s_0 > 0, \quad \text{given}$$

And because of the resource constraint, this economy will have the solution, for every time period and history,

$$c_t(s^t) = y_t(s^t)$$
$$q_t = \beta^t \pi(s^t) u'(y_t(s^t))$$

1.1 Pricing Arrow Securities

Suppose we have an endowment economy with a large number of identical consumer. Each consumer's endowment follows some random markov process. In a sequential trading environment, a consumer solves the following problem:

$$\max_{(c_t(s^t), a_{t+1}(s^t))} \quad \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t|s_0) u(c_t(s^t))$$
$$s.t.$$
$$c_t(s^t) + \sum_{s^{t+1}|s^t} Q_t(s_{t+1}|s^t) a_{t+1}(s^{t+1}) \leq y_t(s^t) + a_t(s^t)$$
$$c_t \geq 0$$
$$a_{t+1} \geq \underline{A}$$
$$a_0(s_0) = 0$$
$$s_0 > 0, \quad \text{given}$$

The first order conditions for consumption and asset purchases of the consumer are

$$\beta^t \pi(s^t) u(c(s^t)) = \mu(s^t)$$
$$Q_t(s_{t+1}|s^t) \mu(s^t) = \sum_{s^{t+1}|s^t} \mu(s^{t+1})$$

Combining, we get

$$Q_t(s_{t+1}|s^t) = \pi(s_{t+1}|s^t) \beta \frac{u'(s^{t+1})}{u'(s^t)}$$

It's also useful to see the relation to the date-0 arrow price. I will change up notation a bit and call that price $q_t(s^t)$. Recall from an Arrow-Debreu market that

$$\beta \pi(s^t) \frac{u'(s^t)}{u'(s_0)} = q_t(s^t)$$

It follows relatively easily, and we have shown this before in similar environments that,

$$Q_t(s_{t+1}|s^t) = \frac{q_{t+1}(s^{t+1})}{q_t(s^t)}$$

The later portion of this equation is often referred to as a stochastic discount factor, m_{t+1}.

$$Q_t(s_{t+1}|s^t) = \pi(s_{t+1}|s^t)m_{t+1}(s^t + 1)$$

1.2 Pricing any Arbitrary Asset

Now consider any arbitrary asset that pays some stream of dividends or payouts over the course of the future $\{d_t(s^t)\}_{t=0}^{\infty}$. This can be any arbitrary function of the state. I am going to use a little bit of new notation. I will use $q_t(s^t)$ as the date 0 price of 1 unit of consumption in any state s^t. This is the typical arrow price that we usually denote with p. Instead, I am going to use $p_\tau^0(s^\tau)$ denote the date 0 price of the asset $\{d_t(s^t)\}_{t=\tau}^{\infty}$, given that history s^τ has been realized. If this is confusing now, we will work through exactly what that means. For instance, the initial price of this asset is simply the sum of all the value that this asset will pay.

$$p_0^0(s_0) = \sum_t \sum_{s^t} q_t(s^t)d_t(s^t)$$

Now let's imagine that some amount of history has transpired and we want the price of the remaining value of the asset, were it to be traded. The date-0 price of trading this asset in period τ, given s^τ would be

$$p_\tau^0(s^\tau) = \sum_{t=\tau}^{\infty} \sum_{s^t|s^\tau} q_t(s^t)d_t(s^t)$$

This is simply the truncated sum of payments. Suppose we care more about the sequential budget and we want to price the value of trading the asset in period τ in terms of consumption in period τ. This would be

$$p_\tau^\tau(s^\tau) = \sum_{t=\tau}^{\infty} \sum_{s^t|s^\tau} \frac{q_t(s^t)}{q_\tau(s^\tau)} d_t(s^t)$$

We simply renormalize the payoffs relative to the price of consumption at τ. Another useful way to characterize the value of the asset is recursively.

$$p_\tau^\tau(s^\tau) = d_\tau(s^\tau) + \sum_{s^{\tau+1}|s^\tau} p_{\tau+1}^\tau(s^{\tau+1})$$

$$p_\tau^\tau(s^\tau) = d_\tau(s^\tau) + \sum_{s^{\tau+1}|s^\tau} \frac{q_{\tau+1}(s^{\tau+1})}{q_\tau(s^\tau)} p_{\tau+1}^{\tau+1}(s^{\tau+1})$$

$$p_\tau^\tau(s^\tau) = d_\tau(s^\tau) + \sum_{s^{\tau+1}|s^\tau} Q_t(s_{t+1}|s^t) p_{\tau+1}^{\tau+1}(s^{\tau+1})$$

$$p_\tau^\tau(s^\tau) = d_\tau(s^\tau) + E_t m_{t+1} p_{\tau+1}^{\tau+1}(s^{\tau+1})$$

2 Lucas Tree Economy

Now we will move to considering a full, albeit simple, equilibrium model of asset pricing. This model is from Lucas 1978, and much of this treatment is from Chapter 13 of Ljungqvist and Sargent. It is often referred to as a Lucas Tree model or a consumption-based asset pricing model.

In this model, there is a large number of identical consumers. The only durable good in the economy is a set of identical "trees," on for each person in the economy. We will consider a representative agent and a single tree. Each period, this tree yields y_t fruit, or dividends. The yield of the tree follows some random markov process. The agent has access to two savings assets, one risk free bond and equity in the tree, e.g. a stock. The one unit of the risk free bond has some price R_t^{-1} and pays out a unit of consumption in the following period. One unit of stock in the tree has price p_t and entitles the agent to y_{t+1}, as well as the value of the stock in following period, p_{t+1}. All together, the household's problem can be described as

$$\begin{aligned} \max_{(c_t, L_t, N_t)} \quad & E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ s.t. & \\ c_t + R_t^{-1} L_t + p_t N_t &\leq L_{t-1} + (p_t + y_t) N_{t-1} \\ c_t, N_t &\geq 0 \\ L_t &\geq \underline{A} \\ L_{-1} &= 0 \\ N_{-1} &= 1 \\ y_0, & \quad \text{given} \end{aligned}$$

There are no firms in this simple economy. In order for markets to clear, the agent must consume the entire endowment, the bonds have to have 0 net supply, and all of the equity in the tree must be owned by the agent.

$$\begin{aligned} c_t &= y_t \\ L_t &= 0 \\ N_t &= 1 \end{aligned}$$

These two financial assets give us two Euler Equations, and associated transversality conditions.

$$\begin{aligned} u'(y_t) R_t^{-1} &= E_t \beta u'(y_{t+1}) \\ u'(y_t) p_t &= E_t \beta (y_{t+1} + p_{t+1}) u'(y_{t+1}) \end{aligned}$$

We could rewrite these equations using the stochastic discount factor formulation.

$$\begin{aligned} R_t^{-1} &= E_t m_{t+1} \\ p_t &= E_t (y_{t+1} + p_{t+1}) m_{t+1} \end{aligned}$$

2.1 Stock Prices

Notice that the Euler equation on stock prices looks a bit like the recursive formulation we worked with before, only with additional stochastic discount factor. Suppose y_t follows a discrete markov chain,

$$p_t = E_t \beta (y_{t+1} + p_{t+1}) \frac{u'(y_{t+1})}{u'(y_t)}$$

$$p_t = \sum \beta \pi(y_{t+1}|y_t) \frac{u'(y_{t+1})}{u'(y_t)} y_{t+1} + \sum \beta \pi(y_{t+1}|y_t) \frac{u'(y_{t+1})}{u'(y_t)} p_{t+1}$$

This shows that again, even though markets are not exactly complete, the arrow security price is appearing again in the pricing of our assets. If we follow out the recursion, we get,

$$p_t = \sum_j \sum_{s^{t+j}|s^t} \beta \pi(y_{t+j}|y_t) \frac{u'(y_{t+j})}{u'(y_t)} y_{t+j} + E_t \lim_{k \to \infty} \beta^k \frac{u'(y_{t+k})}{u'(y_t)} p_{t+k}$$

The last term is 0 by the transversality condition, which gives us that stock prices are equal to the sum of all future dividends, discounted by the stochastic discount factor expression.

2.2 Martingale Theory of Stock Prices

A version of the theory of "efficient stock markets" is sometimes that the price of a stock ought to evolve according to a martingale. For those unfamiliar with the term, a stochastic process $\{x_t\}$ is said to follow a martingale if

$$E[x_{t+1}|x_t, \ldots, x_0] = x_t$$

That is, conditioning on the entire history, the expectation of x_{t+1} is equal to x_t. Consider the Euler equation with the stock price,

$$p_t = E_t \beta (y_{t+1} + p_{t+1}) \frac{u'(y_{t+1})}{u'(y_t)}$$

The right-hand side can be expressed as the product of two expectations of random variables and the covariance between the variables.

$$p_t = \beta E_t (y_{t+1} + p_{t+1}) E_t \frac{u'(y_{t+1})}{u'(y_t)} + \beta \text{cov}_t \big((y_{t+1} + p_{t+1}), \frac{u'(y_{t+1})}{u'(y_t)} \big)$$

In order to get the martingale result, we have to make a few assumptions that are relatively restrictive. First, it must be that $E_t \frac{u'(y_{t+1})}{u'(y_t)}$ is constant, and it must also be that $\text{cov}_t\big((y_{t+1} + p_{t+1}), \frac{u'(y_{t+1})}{u'(y_t)}\big) = 0$. This would be true if agents were risk neutral, which is a pretty strong assumption.

2.3 Crop Insurance

For the sake of this process, let $\{y_t\}$ be a continuous random variable that evolves according to a condition density function $f(y_{t+1}, y_t)$. This is related to the transition function object we studied last week. Suppose we want to price some crop insurance that pays one unit of consumption in the following period, but only if $y_{t+1} \leq \alpha$. This will have a first order condition that is similar to the ones we have already studied, but truncated.

$$q_\alpha(y_t) = \beta \int_0^\alpha \frac{u'(y_{t+1})}{u'(y_t)} f(y_{t+1}, y_t) dy_{t+1}$$

This example can show us some instructive results. For example, consider the case when the agent is risk neutral. This implies that utility is linear, and that marginal utility is constant. Thus, this will break down to

$$q_\alpha(y_t) = \beta \int_0^\alpha f(y_{t+1}, y_t) dy_{t+1}$$
$$q_\alpha(y_t) = \beta \Pr(y_{t+1} \leq \alpha | y_t)$$

or the "actuarial fair" price of insurance.

Consider another example with a risk averse agent who is currently experiencing okay crop yields, i.e. $y_t > \alpha$. Note that this implies for all $y_{t+1} \leq \alpha$,

$$\frac{u'(y_{t+1})}{u'(y_t)} > 1$$

since marginal utility is decreasing for a risk averse agent. Thus, we get the result that an agent will pay a premium for that risk.

$$q_\alpha(y_t) > \beta \int_0^\alpha f(y_{t+1}, y_t) dy_{t+1}$$
$$q_\alpha(y_t) > \beta \Pr(y_{t+1} \leq \alpha | y_t)$$

3 Equity Premium Puzzle

One of the biggest problems in macro finance is referred to as the Equity Premium Puzzle. The "puzzle" amounts to the fact that for parameterizations of CRRA preferences that macro economists consider to be reasonable, there is no way to justify the premium in returns on stocks over bonds. There are a few ways to characterize this problem, one of which uses much of what we have just covered. Consider an asset that has a price of 1 and a risky payoff next period of R_{t+1}. The Euler equation for the household can be written using the stochastic discount factor as

$$1 = E_t m_{t+1} R_{t+1}$$

Again by breaking apart the expectations, we can get

$$1 = E_t m_{t+1} E_t R_{t+1} + \text{cov}_t(m_{t+1}, R_{t+1})$$

The Cauchy-Shwartz Inequality tells us that $|\text{cov}(x, y)| \leq \text{var}(x)\text{var}(y)$. This gives us a bound.

$$\begin{aligned} 1 &\geq E_t m_{t+1} E_t R_{t+1} - \sigma_t(m_{t+1})\sigma_t(R_{t+1}) \\ E_t R_{t+1} &\leq \frac{1}{E_t m_{t+1}} + \frac{\sigma_t(m_{t+1})}{E_t m_{t+1}}\sigma_t(R_{t+1}) \end{aligned}$$

Since we know that the risk free rate is the inverse of the stochastic discount factor from the household first order condition for bonds, we get

$$E_t R_{t+1} \leq R^f_{t+1}(1 + \sigma_t(m_{t+1})\sigma_t(R_{t+1}))$$

If we assume CRRA preferences of $u(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$, then we get that

$$\sigma_t(m_{t+1}) = \sigma_t(\beta(\frac{c_{t+1}}{c_t})^{-\gamma})$$

In traditional settings, we typically find it very hard to believe that $\gamma > 10$ is crazy, but for this equation to justify the kind of difference in expected returns, we would need γ closer to 25. Part of the problem is that aggregate consumption is not very volatile, and so it may point to a problem with the representative agent formulation.