# Economics 8105
# Macroeconomic Theory
# Recitation 7

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Sections adapted from Anh Thu (Monica) Tran Xuan's 8106 Notes and
Sergio Ocampo Diaz's Math Camp Notes

**Outline:**

- Measure Theory
- Markov Processes
- Real Business Cycle Models
- Constant Consumption to Output Ratios
- Chapter 9 Summary

## 1 Some Math Premilinaries

Many of the models that we talk about for the rest of the year will include some feature of random variation. We would like to be able to use the dynamic programming results that we have already applied to deterministic settings to solve these more complex problems. For the most part, these results go through with nearly identical conditions. In this section, I will introduce some basic concepts of measure theory and Markov processes. The technical details are not terribly important, but it is good to be familiar with the vocabulary. You can find in-depth discussion of these subjects in Chapters 7 and 8 of SLP.

### 1.1 Measure Spaces and Measure Functions

So far, we have developed the notion of a metric and a metric space. A metric tells a distance between two general objects, e.g. functions, operations, etc. As you may guess, a measure captures the notion of length, area, volume, etc. The first step in "measuring" anything is to determine what we are able to know.

**Definition 1.1.** Let $X$ be a set and $\mathscr{X} \subseteq 2^X$ be a family of subsets of $X$. $\mathscr{X}$ is called a **$\sigma$-algebra** if

1. $\emptyset, X \in \mathscr{X}$
2. $E \in \mathscr{X} \Rightarrow E^C = X \backslash E \in \mathscr{X}$ ($\mathscr{X}$ is closed under complement.)
3. $\forall n, E_n \in \mathscr{X} \Rightarrow \cup_{n=1}^{\infty} E_n \in \mathscr{X}$ ($\mathscr{X}$ is closed under countable union.)

A $\sigma$-algebra imposes certain consistency to the family of sets under consideration. Only subsets of the $\sigma$-algebra can be known, hence measured. First, it must be possible to know when none or all of the outcomes occurred. Also if there is an outcome that occurred it must be possible to determine if it didn't. Finally if it is possible to determine that some outcomes occurred individually it can also be determined if at least one or all of them were realized.

**Definition 1.2.** For any set $X$ and $\sigma$-algebra $\mathscr{X}$, the pair $(X, \mathscr{X})$ is called a *measurable space*. Any set $E \in \mathscr{X}$ is *measurable*.

**Definition 1.3.** For any metric space $(X, \rho)$, the **Borel algebra** is the smallest $\sigma$-algebra containing the open balls, i.e. containing all sets of the form $E = \{x \in X : \rho(x, x_0) < \delta\}$where $x_0 \in X$ and $\delta > 0$.

**Definition 1.4.** A ***measure*** is a function, $\mu : \mathscr{X} \to \mathbb{R}_+$, that satisfies the following conditions:

1. null set has measure zero; $\mu(\emptyset) = 0$,
2. $\mu(E) \geq 0, \forall E \in \mathscr{X}$
3. $\mu(\cdot)$ is *countably additive*;

$$\mu\left(\bigcup_{i \in I} E_i\right) = \sum_{i \in I} \mu(E_i),$$

for every disjoint countable collection of sets, $I$, in $\mathscr{X}$.

**Definition 1.5.** For any set $X$, $\sigma$-algebra $\mathscr{X}$, and measure $\mu$, the triplet $(X, \mathscr{X}, \mu)$ is called a *measure space*. If $\mu(X) = 1$, then $\mu(\cdot)$ is called a *probability measure*. In this case, we call the triple $(X, \mathscr{X}, \mu)$ a *probability space*.

One can think of a function as mapping certain events in a given measure space to outcomes in another measure space. A function is measurable if the sets that induce a given outcome are measurable.

**Definition 1.6.** Given a measurable space $(X, \mathscr{X})$, a real-valued function $f : X \to \mathbb{R}$ is measurable w.r.t. $\mathscr{X}$ ($\mathscr{X}$-measurable) if

$$\{x \in X | f(x) \leq a\} \subseteq \mathscr{X}, \forall a \in \mathbb{R}$$

**Definition 1.7.** Given measurable spaces $(X, \mathscr{X})$ and $(Y, \mathscr{Y})$. Let $\Gamma : X \rightrightarrows Y$ be a correspondence. Then the function $h : X \to Y$ is a measurable selection from $\Gamma$ if $h$ is measurable and $h(x) \in \Gamma(x), \forall x \in X$.

**Definition 1.8.** Let $(X, \mathscr{X}, \mu)$ be a measure space. A proposition is said to hold *almost everywhere* (a.e.) or *almost surely* (a.s.) if there exists a set $E \in \mathscr{X}$ such that $\mu(E) = 0$ and the proposition holds in $E^c$.

## 1.2 Markov Processes

Nearly every stochastic process that we examine in economics can be characterized as a Markov Process. (i.i.d. is a special case of a Markov process). In words, a markov process is a stochastic process in which the probability of some future event depends only on the current event, rather than the entire history of the process. This is a convenient characteristic to include in models for obvious reasons. The key feature of a Markov process is a transition matrix, or more generally a transition function.

**Definition 1.9.** Let $(X, \mathscr{X})$ be a measurable space. A transition function is a function $Q : X \times \mathscr{X} \to [0, 1]$, such that:

1. for each $x \in X$, $Q(x, \cdot)$ is a probability measure on $(X, \mathscr{X})$,
2. and, for each $E \in \mathscr{X}$, $Q(\cdot, E)$ is a $\mathscr{X}$-measurable function.

**Definition 1.10.** A stationary stochastic process $\{x_t\}$ is a *markov process* if for $n \geq 1$,

$$Pr(x_{t+n} | x_t, x_{t-1}, \ldots, x_0) = Pr(x_{t+n} | x_t)$$

For a markov process with transition function $Q$,

$$Pr(x_{t+1} \in X | x_t) = Q(x_t, X)$$

In a discrete setting, the transition function is often characterized by a transition matrix $P$ where an element of the matrix $p_{ij}$ is such that,

$$Pr(x_{t+1} = j | x_t = i) = p_{ij}$$

# 2 Real Business Cycle Model

One of the legacies of Minnesota Economics is the "Real Business Cycle" model. This type of model has been extended and studied widely since its introduction in the 80s. In this environment, everything will be exactly as we have studied it before. But now we will introduce a stochastic process. The standard way to introduce this process is in the production function.

Our stochastic process will be denoted by $s_t$. For simplicity, lets assume that $s_t$ is discrete. I.e, $s_t \in \{s_0, s_1, \ldots, s_N\}$. I will denote $s^t = (s_t, s_{t-1}, \ldots, s_0)$, the entire sequence up to time $t$. In general, $Pr(s^t) = \pi(s^t)$. We will assume that $s_t$ follows a Markov process. Why is this so important?

A standard production function is $F(k_t, n_t, s_t) = A(s_t)k_t^{\alpha}n_t^{1-\alpha}$. You can think of $A(s_t)$ as some sort of random productivity. Or perhaps something about productivity that we do not observe or understand...

## 2.1 Arrow Debreu Competitive Equilibrium

The definition of competitive equilibrium in this environment is very similar to the deterministic environment we already know and love. However, instead of solving for one deterministic sequence of consumption or capital, we know need to solve for sequences of consumption given every possible evolution of the stochastic shock. We will assume there is one infinitely lived representative agent and one single sector firm.

**Definition 2.1.** An **Arrow-Debreu Equilibrium** is

- an allocation for the HH: $z^H = \{(c_t(s^t), k_{t+1}(s^t), x_t(s^t), n_t(s^t), l_t(s^t))\}_{t=0}^{\infty}$
- an allocation for the firm: $z^F = \{(y_t^f(s^t), k_t^f(s^t), n_t^f(s^t))\}_{t=0}^{\infty}$
- a system of prices: $p = \{(p_t(s^t), w_t(s^t), r_t(s^t))\}_{t=0}^{\infty}$

such that

(HH) Given $p$, $\forall i \in I$, $z^H$ solves

$$\max_{(c_t(s^t),k_{t+1}(s^t),x_t(s^t),n_t(s^t),l_t(s^t))} \quad \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t)u(c_t(s^t),l_t(s^t))$$

$$s.t.$$

$$\sum_{t=0}^{\infty}\sum_{s^t}p_t(s^t)\left[c_t(s^t)+x_t(s^t)\right] \leq \sum_{t=0}^{\infty}\sum_{s^t}\left[w_t(s^t)n_t(s^t)+r_t(s^t)k_t(s^{t-1})\right]$$

$$k_{t+1}(s^t) \leq x_t(s^t)+(1-\delta)k_t(s^{t-1}),\ \forall t,\ s^t$$

$$c_t(s^t),k_{t+1}(s^t),n_t(s^t),l_t(s^t) \geq 0,\ \forall t,\ s^t$$

$$k_0,\ s_0\ > 0,\ \text{ given}$$

(Firm) Given $p$, $z^F$ solves

$$\max_{(y_t^f(s^t),k_t^f(s^t),n_t^f(s^t))} \sum_{t=0}^{\infty}\sum_{s^t}\left[p_t(s^t)y_t^f(s^t)-w_t(s^t)n_t^f(s^t)-r_t(s^t)k_t^f(s^t)\right]$$

$$s.t.$$

$$y_t^f(s^t) \leq F(k_t^f(s^t),n_t^f(s^t),s_t),\ \forall t,\ s^t$$

$$k_t^f(s^t),n_t^f(s^t),y_t^f(s^t) \geq 0, \forall t$$

(Mkt) For all $t$, $s^t$,

$$\text{(Goods Market)} \quad c_t(s^t)+x_t(s^t)=y_t^f(s^t)$$

$$\text{(Labor Market)} \quad n_t(s^t)=n_t^f(s^t)$$

$$\text{(Capital Market)} \quad k_t(s^{t-1})=k_t^f(s^t)$$

The first order conditions of this model give us,

$$\beta^t\pi(s^t)u_c(s^t)=p_t(s^t)\lambda \tag{1}$$

$$\beta^t\pi(s^t)u_l(s^t)=w_t(s^t)\lambda \tag{2}$$

$$p_t(s^t)=\sum_{s_{t+1},s^t}r_{t+1}(s^{t+1})+(1-\delta)p_{t+1}(s^{t+1}) \tag{3}$$

$$r_t(s^t)=p_t(s^t)F_k(s^t) \tag{4}$$

$$w_t(s^t)=p_t(s^t)F_n(s^t) \tag{5}$$

The first order condition for consumption gives us the expression for an Arrow price in this stochastic environment. Notice that the price will include the probability of the state of the world, similar to how the price bakes in the present value discounting.

$$p_t(s^t) = \beta^t \pi(s^t) \frac{u_c(s^t)}{u_c(s_0)}$$

Writing the Euler equation in a stochastic environment is a little different, since time no longer advances on a deterministic path.

$$\begin{aligned}
\beta^t \pi(s^t) u_c(s^t) &= p_t(s^t)\lambda \\
\beta^t \pi(s^t) u_c(s^t) &= \lambda \sum_{s_{t+1}} r_{t+1}(s^{t+1}) + (1-\delta) p_{t+1}(s^{t+1}) \\
\beta^t \pi(s^t) u_c(s^t) &= \sum_{s_{t+1}} \lambda p_{t+1}(s^{t+1}) \big( \frac{r_{t+1}(s^{t+1})}{p_{t+1}(s^{t+1})} + (1-\delta) \big) \\
\beta^t \pi(s^t) u_c(s^t) &= \sum_{s_{t+1}} \beta^{t+1} \pi(s^{t+1}) u_c(s^{t+1}) \big( \frac{r_{t+1}(s^{t+1})}{p_{t+1}(s^{t+1})} + (1-\delta) \big) \\
\beta^t \pi(s^t) u_c(s^t) &= \sum_{s_{t+1}} \beta^{t+1} \pi(s^{t+1}) u_c(s^{t+1}) \big( F_k(s^{t+1}) + (1-\delta) \big) \\
u_c(s^t) &= \beta \sum_{s_{t+1}} \frac{\pi(s^{t+1})}{\pi(s^t)} u_c(s^{t+1}) \big( F_k(s^{t+1}) + (1-\delta) \big) \\
u_c(s^t) &= \beta \sum_{s_{t+1}} \pi(s_{t+1}|s^t) u_c(s^{t+1}) \big( F_k(s^{t+1}) + (1-\delta) \big) \\
u_c(s^t) &= \beta E[u_c(s^{t+1}) \big( F_k(s^{t+1}) + (1-\delta) \big) | s^t]
\end{aligned}$$

The transition in probabilities and the summation in the second to last line comes from

$$\begin{aligned}
\frac{\pi(s^{t+1})}{\pi(s^t)} &= \frac{\pi(s^{t+1}|s^t)\pi(s^t)}{\pi(s^t)} \\
&= \pi((s_{t+1}, s^t)|s^t) \\
&= \pi(s_{t+1}|s^t)
\end{aligned}$$

## 2.2 Sequential Markets Equilibrium

In a stochastic setting, the Arrow Debreu construction automatically assumes complete insurance. This means that the agent can borrow across different states of the world and insure herself to any kind of risk. This is implicit in the summations across all states in the Arrow Debreu setup. In a sequential market equilibrium, this takes the form of Arrow securities. I won't write out the whole equilibrium definition again, but the household budget constraint will become

$$c_t(s^t) + x_t(s^t) + \sum_{s_{t+1}} q_t(s_{t+1}, s^t) a_{t+1}(s_{t+1}, s^t) \leq w_t(s^t) n_t(s^t) + r_t(s^t) k_t(s^{t-1}) + a_t(s^t)$$

In the deterministic setting, the agent bought one bond for the next period. Now, the agent buys an asset for every possible realization of the stochastic shock. The assets are thus priced accordingly. If $\mu_t(s^t)$ is the lagrange multiplier on the budget constraint, The first order conditions for these assets give us

$$\begin{aligned} q_t(s_{t+1}, s^t)\mu_t(s^t) &= \mu_{t+1}(s^{t+1}) \\ q_t(s_{t+1}, s^t)\beta^t\pi(s^t)u_c(s^t) &= \beta^{t+1}\pi(s^{t+1})u_c(s^{t+1}) \\ q_t(s_{t+1}, s^t) &= \beta\pi(s_{t+1}|s^t)\frac{u_c(s^{t+1})}{u_c(s^t)} \end{aligned}$$

This formulation will be used frequently later in the course, when we are interested in the types of assets that agents can use to save. Just as in the deterministic case, we will have a borrowing limit here for every state and time period. What might happen if we imagine a borrowing limit that could potentially be binding? If the markets were incomplete, and the agent only had access to one asset for every state of the world, what would the price of the asset become?

### 2.3 The Social Planner Problem

Just as in the deterministic setting, it will be useful to rewrite the competitive equilibrium as a dynamic planning problem. In this section, I will drop the investment and leisure allocations. We will be assuming all the typical assumptions about production and utility that imply all the constraints are binding and allow us to make the appropriate substitutions. The sequential social planner problem is

$$\begin{aligned} v(k_0, s_0) = \max_{(c_t(s^t), k_{t+1}(s^t), n_t(s^t),} & \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t)u(c_t(s^t), 1-n_t(s^t)) \\ & s.t. \\ c_t(s^t) + k_{t+1}(s^t) &\leq F(k_t(s^{t-1}), n_t(s^t), s_t) + (1-\delta)k_t(s^{t-1}) \; \forall t, \; s^t \\ c_t(s^t), k_{t+1}(s^t), n_t(s^t) &\geq 0, \; \forall t, \; s^t \\ n_t(s^t) &\leq 1, \; \forall t, \; s^t \\ k_0, \; s_0 &> 0, \;\; \text{given} \end{aligned}$$

# 3 Dynamic Programming - A Special Case

Now we turn to our favorite tool, dynamic programming. Chapter 9 of Stokey Lucas Prescott extends the deterministic results from Chapter 4 to a stochastic setting. In the next section, I have borrowed an abbreviation of the assumptions and theorems in Chapter 9 from Monica

Tran Xuan's notes—She was the TA for this course last year. You can find her full treatment of this material on her website under Recitation 2 for 8106. In this section we are going to walk through a proof for a special case. In the A(k) model with constant-relative-risk-aversion preferences, we can prove that consumption and investment will be constant shares of output. This is a useful result for solving problems with stochastic taxation.

$$u(c,l) = \frac{c^{1-\sigma}}{1-\sigma}$$
$$F(k,n,s) = A(s)k$$

I will write the problem as though $\delta = 1$. However, in the A(k) model, you can simply imagine that the productivity term $A(s)$ already includes depreciation. We will also assume that the stochastic process is a stationary Markov process. In this setting, the dynamic social planners problem becomes

$$v(k,s) = \max_{k' \in \Gamma(k,s)} \frac{[A(s)k - k']^{1-\sigma}}{1-\sigma} + \beta E[V(k',s')|s]$$
$$\Gamma(k,s) = \{k' | 0 \le k' \le A(s)k\}$$

**Proposition 3.1.** *In this environment, the value function $V$ is homogeneous of degree $1-\sigma$ in capital and the policy function $g_k$ is homogeneous of degree 1 in capital.*

*Proof.* The proof for this is identical to problem 6 from Problem set 2. It is much easier to prove when you write the problem sequentially, as in section 2.3. First you show using a proof by contradiction that the policy functions must be homogeneous of degree 1, since the feasibility constraints are linear in initial capital. Then the homogeneity of the value function will follow easily. Through the theorems of Chapter 9, these two problems are equivalent. □

**Proposition 3.2.** *In this environment, if the stochastic process for $s_t$ is i.i.d., then the policy functions for capital and consumption, $g_k$ and $g_c$, can be written as constant fractions of output, where for some $\phi \in [0,1]$,*

$$g_k(k,s) = \phi A(s)k$$
$$g_c(k,s) = (1-\phi)A(s)k$$

*Proof.* Using proposition 3.1, we can rewrite the objective function of the problem,

$$v(k,s) = \max_{k' \in \Gamma(k,s)} \frac{(A(s)k)^{1-\sigma}[1 - \frac{k'}{A(s)k}]^{1-\sigma}}{1-\sigma} + \beta E[k'^{1-\sigma} V(1,s')|s]$$

$$v(k,s) = \max_{k' \in \Gamma(k,s)} (A(s)k)^{1-\sigma} \left( \frac{[1 - \frac{k'}{A(s)k}]^{1-\sigma}}{1-\sigma} + \beta (\frac{k'}{A(s)k})^{1-\sigma} E[V(1,s')|s] \right)$$

$$v(k,s) = (A(s)k)^{1-\sigma} \left( \max_{k' \in \Gamma(k,s)} \frac{[1 - \frac{k'}{A(s)k}]^{1-\sigma}}{1-\sigma} + \beta (\frac{k'}{A(s)k})^{1-\sigma} E[V(1,s')|s] \right)$$

We can rewrite the feasibility correspondence

$$\Gamma(k,s) = \{k'|0 \le k' \le A(s)k\}$$
$$\Gamma(k,s) = \{k'|0 \le \frac{k'}{A(s)k} \le 1\}$$

For posterity, let's consider the case when $s_t$ follows a general Markov process. Then we can define $D(s) = E[V(1,s')|s]$. The problem becomes,

$$v(k,s) = (A(s)k)^{1-\sigma}\left(\max_{\phi\in[0,1]} \frac{[1-\phi]^{1-\sigma}}{1-\sigma} + \beta\phi^{1-\sigma}D(s)\right)$$
$$g_k(k,s) = \phi(s)A(s)k$$

As the maximization problem does not have $k$ in it, it is evident that the optimal $\phi$ does not depend on $k$. However, it may still depend on $s$. If we use the additional assumption that $s$ is i.i.d., then $E[V(1,s')|s] = E[V(1,s')]$, which we can simply write as a constant $D$. Then the problem becomes,

$$v(k,s) = (A(s)k)^{1-\sigma}\left(\max_{\phi\in[0,1]} \frac{[1-\phi]^{1-\sigma}}{1-\sigma} + \beta\phi^{1-\sigma}D\right)$$
$$g_k(k,s) = \phi A(s)k$$

Now, the maximization problem only depends on constants. If we assume that there is an interior solution,

$$[1-\phi]^{-\sigma} = \beta(1-\sigma)\phi^{-\sigma}D$$
$$1-\phi = (\beta(1-\sigma)D)^{-\frac{1}{\sigma}}\phi$$
$$\phi = \frac{1}{1+(\beta(1-\sigma)D)^{-\frac{1}{\sigma}}}$$

Thus, $\phi$ is independent of the state, and the optimal level of consumption and next periods capital is a constant fraction of output. □

# 4 Notes on SLP 9.1 and 9.2

The following notes come to you courtesy of Monica Tran Xuan.

## 4.1 9.1 Principle of Optimality

Suppose $(X,\mathscr{X})$ and $(Z,\mathscr{Z})$ are measurable spaces, and $(S,\mathscr{S}) = (X\times Z, \mathscr{X}\times\mathscr{Z})$ is the product space. $S$ is the set of states for this system, whose elements consist of an exogenous state, $z \in Z$, and an endogenous state, $x \in X$. We will assume that the exogenous state follows a stationary Markov process, with transition function $Q(\cdot)$, defined on $(Z, \mathscr{Z})$.

Given today's state, $s$, let us denote the set of feasible next period states by $\Gamma(s)$; $\Gamma : S \to X$ is the feasibility correspondence. Let $A$ be the graph of $\Gamma(\cdot)$:

$$A = \{(x, y, z) \in X \times X \times Z \mid y \in \Gamma(x, z)\}.$$

Suppose $F : A \to \mathbb{R}$ is one-period payoff function, and $\beta \geq 0$ is the discount factor.

Recall that, in a sequential setting, a planner's problem in each period is to choose a sequence of contingent plans that maximize the discounted expected value of utility. These contingent plans can be viewed as functions of the history of shocks up to a given date $t$, namely, $z^t$:

$$\begin{aligned} \max_{\{x_{t+1}(z^t)\}_{t\geq 0}^{z^t \in Z^t}} \quad & E_0 \sum_{t=0}^{\infty} \beta^t F\left[x_t\left(z^{t-1}\right), x_{t+1}\left(z_t, z^{t-1}\right), z_t\right] \\ s.t. \quad & 0 \leq x_{t+1}\left(z_t, z^{t-1}\right) \leq \Gamma\left[x_t(z^{t-1}), z_t\right] \text{ for all } z^{t-1} \text{ and } z_t, \\ & \text{given } k_0 \text{ and } z_0. \end{aligned} \tag{6}$$

The recursive counterpart of this optimization problem can be written in terms of the following functional equation:

$$v(s) = v(x, z) = \sup_{y \in \Gamma(s)} \left\{ F(x, y, z) + \beta \int_Z v(y, z') Q(z, dz') \right\}. \tag{7}$$

To generalize this problem, to a setting where $z$ is a Markov process instead of a Markov chain, let $(Z^t, \mathscr{Z}^t)$ be the product space up to period $t \geq 1$, and $z^t \in Z^t$ be the partial history of shocks up to date $t$. Then, we can define a plan as follows:
**Definition 4.1.** A plan is a value $\pi_0 \in X$ and a sequence of measurable functions $\pi_t : Z^t \to X$, for $t = 1, 2, \ldots$.
**Definition 4.2.** A plan $\pi$ is feasible from $s_0 \in S$, if:

1. $\pi_0 \in \Gamma(s_0)$,
2. and, $\pi_t(z^t) \in \Gamma[\pi_{t-1}(z^{t-1}), z_t]$, for all $z^t \in Z^t$ and $t = 1, 2, \ldots$.

Let us denote the set of all feasible plans from $s_0$ by $\Pi(s_0)$. Recall from the deterministic case that, our first requirement for the sequence problem to be well defined was for $\Gamma(\cdot)$ to be a non-empty correspondence. Here, we need a stronger assumption that ensures the existence of *measurable functions*. This is done in the following assumptions.
**Assumption.** ***9.1***

*$\Gamma(\cdot)$ is non-empty valued, and $A$ is $(\mathscr{X} \times \mathscr{X} \times \mathscr{Z})$-measurable. Moreover, $\Gamma(\cdot)$ has a measurable selection; i.e. there exists a measurable function $h : S \to X$ such that $h(s) \in \Gamma(s)$ for all $s \in S$.*

It is straightforward to show that, under Assumption 9.1, $\Pi(s_0)$ is non-empty for all $s_0 \in S$.

Next, we have to specify how the planner calculates the expectations in 6. To do so, given the transition function of the Markov process governing $z$, it is straightforward (at least intuitively) to construct a probability measure explaining the evolution of history up to date $t$, namely $\mu^t(z_0, \cdot) : \mathscr{Z}^t \to [0,1]$. Now, if we define a $\sigma$-algebra, $\mathscr{A}$ as:

$$\mathscr{A} = \{C \in \mathscr{X} \times \mathscr{X} \times \mathscr{Z} \mid C \subset A\},$$

for the discounted expected payoff to be well-defined a the general version of 6, we can impose the following assumption:
**Assumption.** ***9.2*** *$F : A \to \mathbb{R}$ is $\mathscr{A}$-measurable, and one of the followings holds:*

1. *$F \geq 0$ or $F \leq 0$.*
2. *For each $s_0 = (x_0, z_0) \in S$ and each plan $\pi \in \Pi(s_0)$, $F[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t]$ is $\mu^t(z_0, \cdot)$-integrable, for $t = 1, 2, \ldots,$ and the limit*

   $$F(x_0, \pi_0, z_0) + \lim_{n\to\infty} \sum_{t=1}^{n} \int_{Z^t} \beta^t F\left[\pi_{t-1}\left(z^{t-1}\right), \pi_t\left(z^t\right), z_t\right] \mu^t\left(z_0, dz^t\right)$$

   *exists (although it might not be bounded).*

Notice that, under Assumptions 9.1 and 9.2, discounted expected payoff (in a planner's problem) is well-defined, and we can define the value of a planner as:

$$v^*(s) = \sup_{\pi\in\Pi(s)} \left\{F(x_0, \pi_0, z_0) + \lim_{n\to\infty} \sum_{t=1}^{n} \int_{Z^t} \beta^t F\left[\pi_{t-1}\left(z^{t-1}\right), \pi_t\left(z^t\right), z_t\right] \mu^t\left(z_0, dz^t\right)\right\}. \quad (8)$$

**Definition 4.3.** If there exists a function $v(\cdot)$ that solves this functional equation, then, we can define the associated policy correspondence as:

$$G(x,z) = \left\{y \in \Gamma(x,z) \mid v(x,z) = F(x,y,z) + \beta \int_Z v(y,z')\, Q(z,dz')\right\}. \quad (9)$$

How can we generate a plan from this correspondence? Notice that, now, $G(x,z)$ being non-empty is not enough to ensure the existence of a feasible plan; now we need existence of a measurable selection from $G(\cdot)$. If such a measurable selection exists, say $g_0, g_1, \ldots$ is a sequence of measurable selections from $G(\cdot)$, then we can generate a plane as:

$$\pi_0 = g_0(s_0),$$
$$\pi_t\left(z^t\right) = g_t\left[\pi_{t-1}\left(z^{t-1}\right), z_t\right], \text{ for all } z^t \in Z^t \text{ and } t \geq 1.$$

Now, we are ready to present a partial counterpart of the principle of optimality for the stochastic setting, analogous to Theorems 4.3 and 4.5:

**Theorem.** ***9.2 (Principle of Optimality–I)***

*Suppose $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, $Q$, $\Gamma$, $F$, and $\beta$ satisfy Assumptions 9.1 and 9.2. Let $v^*(\cdot)$ be the solution to 8, and $v(\cdot)$ the solution to 7, so that:*

$$\lim_{t\to\infty}\int_{Z^t}\beta^t v\left[\pi_{t-1}\left(z^{t-1}\right), z_t\right]\mu^t\left(z_0, dz^t\right) = 0,$$

*for all $\pi \in \Pi(s_0)$, and all $s_0 \in S$. Let $G(\cdot)$ be the correspondence defined by 9, which is non-empty and permits a measurable selection. Then $v^* = v$, and the plan generated by $G(\cdot)$ attains the supremum in 8.*

For the partial converse to Theorem 9.2, we need to strengthen Assumption 9.2.
**Assumption.** ***9.3*** *If $F$ takes on both signs, there is a collection of nonnegative, measurable functions $L_t : S \to \mathbb{R}_+$, $t = 0, 1, \ldots$, such that for all $\pi \in \Pi(s_0)$ and all $s_0 \in S$*

$$|F(x_0, \pi_0, z_0)| \le F_0(s_0);$$
$$|F[pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t]| \le L_t(s_0),\ \textit{all } z_t \in Z^t,\ t = 1, 2, \ldots$$

*and*

$$\sum_{t=0}^{\infty}\beta^t L_t(s_0) < \infty$$

**Theorem.** ***9.4 (Principle of Optimality–II)***

*Suppose $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, $Q$, $\Gamma$, $F$, and $\beta$ satisfy Assumptions 9.1 through 9.3. Let $v^*(\cdot)$ be the solution to (8). Assume that $v^*$ is measurable and satisfies (7), and define $G$ by (9). Assume that $G$ is nonempty and permits a measurable selection. Let $(x_0, z_0) = s_0 \in S$, and let $\pi^* \in \Pi(s_0)$ be a plan that attains the supremum in (8) for initial condition $s_0$. Then there exists a plan $\pi^G$ generated by $G$ from $s_0$ such that*

$$\pi_0^G = \pi_0^*,\ \textit{and}$$
$$\pi_t^G(s^t) = \pi_t^*(z^t),\ \mu^t(z_0, \cdot)\text{-}a.e.,\ t = 1, 2, \ldots$$

## 4.2 Bounded Returns

Next step, is to ensure the existence of a solution to the functional equation. Like the deterministic case, when the return function is bounded, there is a good chance that this is the case. In this section, we consider the fairly general assumptions under which, this is the case, by focusing on the case of bounded returns.
**Assumption.** ***9.4*** *(analog of A 4.3)*

*$X$ is a convex Borel set in $\mathbb{R}^l$, and $\mathscr{X}$ is its Borel subsets.*

**Assumption.** ***9.5***

*One of the followings holds:*

1. *$Z$ is a countable set, and $\mathscr{Z}$ is the $\sigma$-algebra containing all of its subsets.*
2. *$Z$ is a compact Borel set in $\mathbb{R}^k$, with its Borel subsets $\mathscr{Z}$, and the transition function $Q(\cdot)$ has the Feller property (SLP Chapter 8.1).*

The key role of Assumption 9.5 is to ensure that the integral in 7,

$$Mf(y,z) = \int_Z v(y,z')\, Q(z,dz'), \text{ for all } (y,z) \in X \times Z, \tag{10}$$

maps a bounded continuous function $v : X \times Z \to \mathbb{R}$ into the space of bounded continuous functions over $X \times Z$. Moreover, by Lemma 9.5, assumptions 9.4 and 9.5 ensure that, if $v(\cdot)$ is increasing or concave, then the integral would be an increasing or concave function of $(y,z)$.

Given this property, the rest is quite similar to the stochastic case; first, we may use Blackwell's sufficient conditions to ensure that the mapping defined by 7 is a contraction, and then use the Contraction Mapping Theorem to ensure the existence of a fixed point. First we need the following two assumptions:

**Assumption.** ***9.6*** *(analog of A 4.3)*

*The correspondence $\Gamma : X \times Z \to X$ is non-empty, compact-valued, and continuous.*

**Assumption.** ***9.7****(analog of A 4.4)*

*The function $F : A \to \mathbb{R}$ is bounded and continuous, and $\beta \in (0,1)$.*

Now, we have the following theorem:

**Theorem.** ***9.6 Existence & Uniqueness***

*Under Assumptions 9.4-9.7, the operator $T$, defined by*

$$Tf(x,z) = \sup_{y \in \Gamma(s)} \left\{ F(x,y,z) + \beta \int_Z f(y,z')\, Q(z,dz') \right\}, \tag{11}$$

*maps the set of bounded continuous functions, $C(S)$, into itself, and has a unique fixed point in this set, $v(\cdot) \in C(S)$. Moreover, for all $v_0(\cdot) \in C(S)$:*

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|, \quad n = 1, 2, \ldots.$$

*In addition, the correspondence $G(\cdot)$ defined by 9 is non-empty, compact-valued, and upper hemi-continuous.*

## 4.3 Inheriting Properties of the Value Function

If the operator $M$ in 10 preserves the monotonicity and concavity of the integrand, it is natural to expect that the value function inherits these properties from the payoff function;

what we had in the deterministic case, as well. To formalize this idea, let us introduce the following assumptions. Note that, $A_i$ denotes the $i$-section of the set $A$, in what follows.

**Assumption.** ***9.8***

*For each* $(y, z) \in X \times Z$, $F(\cdot, y, z) : A_{yz} \to \mathbb{R}$ *is strictly increasing.*

**Assumption.** ***9.9***

*For each* $z \in Z$, $x \leq x'$ *implies* $\Gamma(x, z) \in \Gamma(x', z)$.

Now, we have our first inheritance property of the value function:

**Theorem.** ***9.7*** *(analog of Thm 4.7)*

*Under Assumptions 9.4-9.9, the fixed point of operator $T$ in 11 is strictly increasing in $x$, for each $z \in Z$.*

The value function inherits the concavity of the payoff function as well:

**Assumption.** ***9.10***

*For each* $z \in Z$, $F(\cdot, \cdot, z) : A_z \to \mathbb{R}$ *is strictly concave in* $(x, y)$.

**Assumption.** ***9.11***

*The set $A_z$ is convex.*

**Theorem.** ***9.8*** *(analog of Thm 4.8)*

*Under Assumptions 9.4-9.7 and 9.10-9.11, the fixed point of operator $T$ in 11 is strictly concave in $x$, for each $z \in Z$, and the corresponding policy correspondence is a continuous function.*

Finally, $v(\cdot)$ inherits the differentiability of the payoff function, too:

**Assumption.** ***9.12***

*For a fixed* $z \in Z$, $F(\cdot, \cdot, z)$ *is continuously differentiable in* $(x, y)$, *in the interior of* $A_z$.

**Theorem.** ***9.10*** *(analog of Thm 4.11)*

*Suppose Assumptions 9.4-9.7 and 9.10-9.12 hold, $v(\cdot) \in C(S)$ is the fixed point of operator $T$ in 11, and $g : S \to X$ is the corresponding value function. If $x_0 \in intX$, and $g(x_0, z_0) \in int\Gamma(x_0, z_0)$, then $v(\cdot, z_0)$ is continuously differentiable in $x$ at $x_0$, with derivatives given by*

$$v_i(x_0, z_0) = F_i[x_0, g(x_0, z_0), z_0], \text{ for } i = 1, 2, \ldots, l.$$