Economics 8105
Macroeconomic Theory
Recitation 6

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Outline:

- Optimal Taxation with Government Investment

1 Government Expenditure in Production

In these notes we will examine a model in which government expenditure is used in production. In a typical tax-distorted environment, government expenditure is "thrown into the ocean." The government takes away from the total resources and does nothing to the economy other than levy possibly distortionary taxes. In this model, the government good is a type of investment that can be used in government production.

1.1 Optimal Government Spending

Imagine a social planner is free to choose government purchases without needing to worry about funding those purchases. We can think of government purchases as government owned capital or infrastructure. The social planner simply wants to maximize utility subject to the resource constraint. Lets assume that utility is given by constant relative risk-aversion (CRRA) preferences and the production function is cobb-douglas in capital and government investment. Assume that both types of investment have full depreciation.

$$\max_{c_t, k_{t+1}, g_{t+1}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$\begin{aligned} &\text{such that} \\ c_t + k_{t+1} + g_{t+1} &\leq A k_t^{\alpha} g_t^{1-\alpha} \\ c_t, \ k_{t+1}, \ g_{t+1} &\geq 0 \\ k_0, g_0 &> 0 \text{ given} \end{aligned}$$

Let μ_t be the lagrange multiplier on the resource constraint. The first order conditions are given by

$$\beta^t c_t^{-\sigma} = \mu_t \tag{c_t}$$

$$\mu_t = \mu_{t+1}(1-\alpha) A k_{t+1}^{\alpha} g_{t+1}^{-\alpha} \tag{g_{t+1}}$$

$$\mu_t = \mu_{t+1} \alpha A k_{t+1}^{\alpha-1} g_{t+1}^{1-\alpha} \tag{k_{t+1}}$$

Notice that,

$$\begin{aligned} \frac{\mu_t}{\mu_{t+1}} &= (1-\alpha) A k_{t+1}^{\alpha} g_{t+1}^{-\alpha} \\ \frac{\mu_t}{\mu_{t+1}} &= \alpha A k_{t+1}^{\alpha-1} g_{t+1}^{1-\alpha} \\ (1-\alpha) A k_{t+1}^{\alpha} g_{t+1}^{-\alpha} &= \alpha A k_{t+1}^{\alpha-1} g_{t+1}^{1-\alpha} \\ g_t &= \frac{1-\alpha}{\alpha} k_t \end{aligned}$$

If the social planner doesn't have any constraints on taxation, the optimal level of government expenditure will be $g_t = \frac{1-\alpha}{\alpha}$.

1.2 Funding the Expenditures through a TDCE

Question. *Can we implement this optimal solution through a TDCE?*

If the government is able to raise money through lump sum taxes, the solution to the social planner's problem defined in section 1.1 can be obtained through a competitive environment. You can demonstrate this to yourself by showing that the problems are equivalent.

A more interesting case is when the government is only able to raise money through distortionary taxes. Let's examine a case in which the government must fund its expenditures through taxes on capital. We will assume that the firm's take the government investment as given when making their own decisions. Since government purchases are like investment goods, the government will take g_0 as given, similar to the household taking k_0 as given.

A **Tax Distorted Competitive Equilibrium** in this environment is

- an allocation for the HH: $z^H = \{(c_t, k_{t+1}, x_t)\}_{t=0}^{\infty}$
- an allocation for the firm: $z^F = \{(y_t^f, k_t^f)\}_{t=0}^{\infty}$
- a system of prices: $p = \{(p_t, r_t)\}_{t=0}^{\infty}$
- a government policy: $g = \{(g_{t+1}, \tau_t)\}_{t=0}^{\infty}$, such that

(HH) Given p and g, z^H solves

$$\begin{aligned}
\max_{c_t, k_t, x_t} \quad & \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\
& s.t. \\
\sum_{t=0}^{\infty} p_t c_t + p_t x_t &\leq \sum_{t=0}^{\infty} r_t (1-\tau_t) k_t + \pi_t \\
k_{t+1} &\leq x_t, \; \forall t \\
c_t, k_{t+1} &\geq 0, \; \forall t \\
k_0 &> 0, \quad \text{given}
\end{aligned}$$

(Firm) Given p and g, z^F solves

$$\begin{aligned}
\max_{\{(y_t^f, k_t^f)\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \left[p_t y_t^f - r_t k_t^f \right] \\
& s.t. \\
y_t^f &\leq A k_t^{f\alpha} g_t^{1-\alpha}, \; \forall t \\
k_t^f, \; y_t^f &\geq 0, \forall t
\end{aligned}$$

(Mkt) For all t,

$$\begin{aligned}
\text{(Goods Market)} \quad & c_t + x_t + g_{t+1} = y_t^f \leq A k_t^{f\alpha} g_t^{1-\alpha} \\
\text{(Capital Market)} \quad & k_t = k_t^f
\end{aligned}$$

(Govt)

$$\sum_{t=0}^{\infty} p_t g_{t+1} = \sum_{t=0}^{\infty} r_t \tau_t k_t$$

(Profit)

$$\pi_t = y_t^f - r_t k_t^f$$

Notice that in this problem, the firm makes a profit in every period. The production function still has constant returns to scale in its inputs, but the firm receives one of its inputs for free. We can solve the firms problem to get a characterization of profits.

$$\begin{aligned}
r_t &= p_t \alpha A k_t^{\alpha-1} g_t^{1-\alpha} \\
\pi_t &= p_t A k_t^{\alpha} g_t^{1-\alpha} - r_t k_t \\
\pi_t &= p_t A k_t^{\alpha} g_t^{1-\alpha} - p_t \frac{r_t}{p_t} k_t \\
\pi_t &= p_t A k_t^{\alpha} g_t^{1-\alpha} - p_t \alpha A k_t^{\alpha} g_t^{1-\alpha} \\
\pi_t &= p_t (1-\alpha) A k_t^{\alpha} g_t^{1-\alpha} \qquad (1)
\end{aligned}$$

This is a well known result of cobb-douglas production functions. The firm will divide its revenue among its inputs according to the elasticity terms α and $1-\alpha$. The equilibrium of this can be characterized by

$$\begin{aligned}
\beta^t c_t^{-\sigma} &= \lambda p_t \qquad (2) \\
p_t &= r_{t+1}(1-\tau_{t+1}) \qquad (3) \\
r_t &= p_t \alpha A k_{t+1}^{\alpha-1} g_{t+1}^{1-\alpha} \qquad (4) \\
c_t + k_{t+1} + g_{t+1} &= A k_t^{\alpha} g_t^{1-\alpha} \qquad (5)
\end{aligned}$$

in addition to the government budget, the transversality condition, and the initial conditions for g_0 and k_0. Note that we could use the HH budget instead of the government budget, since one implies the other given everything else we know. Combining these equations, we can characterize the Euler Equation.

$$\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma} = \beta(1-\tau_{t+1}) \alpha A k_{t+1}^{\alpha-1} g_{t+1}^{1-\alpha} \qquad (6)$$

If we assume that this economy will converge to steady state, then $\tau_t \to \tau_\infty,\ c_t \to c_\infty,\ k_t \to k_\infty, g_t \to g_\infty$. In the limit,

$$1 = \beta(1-\tau_\infty) \alpha A k_\infty^{\alpha-1} g_\infty^{1-\alpha} \qquad (7)$$

1.3 Building The Ramsey Problem

Now we want to think the best sequence of taxes on capital in order to fund the stream of g_t given by section 1.1. The first set up setting up the Ramsey problem is deriving the implementability constraint, which constrains the set of allocations to those that can be achieved through the tax structure that we have specified. We start with the present value household budget constraint.

$$\sum_{t=0}^{\infty} p_t c_t + p_t k_{t+1} \leq \sum_{t=0}^{\infty} r_t(1-\tau_t)k_t + \pi_t$$
$$\sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} r_t(1-\tau_t)k_t - p_t k_{t+1} + \pi_t$$
$$\sum_{t=0}^{\infty} p_t c_t \leq r_0(1-\tau_0)k_0 + \sum_{t=0}^{\infty} r_{t+1}(1-\tau_{t+1})k_{t+1} - p_t k_{t+1} + \pi_t$$

Using the no arbitrage condition in (3), we can see that the sum term on right hand side is a telescoping sum, as $r_{t+1}(1-\tau_{t+1}) - p_t = 0$. Thus, we have

$$\sum_{t=0}^{\infty} p_t c_t \leq r_0(1-\tau_0)k_0 + \lim_{T\to\infty} p_T k_{T+1} + \sum_{t=0}^{\infty} \pi_t$$
$$\sum_{t=0}^{\infty} p_t c_t \leq r_0(1-\tau_0)k_0 + \sum_{t=0}^{\infty} \pi_t \tag{8}$$

where the second step comes from the transversality condition. Now we need to write (8) in terms of allocations so that we can include it in a planner's problem. Additionally, note that τ_0 is a lump sum tax on the initial endowment of the household. Any optimal tax scheme will tax this initial endowment as much as possible as it does not distort any future decisions, so we can set $\tau_0 = 1$ for simplicity.

Using the first order condition for consumption, (2), in period t and the initial period with $p_0 = 1$, we can get an expression for price:

$$\beta^t \frac{c_t^{-\sigma}}{c_0^{-\sigma}} = p_t \tag{9}$$

Substituting (9) and (1) into (8), we get the implementability constraint in terms of only allocations:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{-\sigma}}{c_0^{-\sigma}} c_t \leq \sum_{t=0}^{\infty} \beta^t \frac{c_t^{-\sigma}}{c_0^{-\sigma}} (1-\alpha)Ak_t^{\alpha} g_t^{1-\alpha}$$
$$\sum_{t=0}^{\infty} \beta^t c_t^{-\sigma}(c_t - (1-\alpha)Ak_t^{\alpha} g_t^{1-\alpha}) \leq 0$$

Now we can set up the Ramsey problem. In general, a Ramsey problem takes a stream of government expenditures as given as maximizes utility over the possible ways to fund that expenditure. In this case, it would be feasible to ask the Ramsey problem to solve for the optimal level of government expenditure as well.

$$\max_{c_t,k_{t+1},g_{t+1}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

such that

$$\sum_{t=0}^{\infty} \beta^t c_t^{-\sigma}(c_t - (1-\alpha)Ak_t^{\alpha}g_t^{1-\alpha}) \leq 0 \tag{10}$$

$$c_t + k_{t+1} + g_{t+1} \leq Ak_t^{\alpha}g_t^{1-\alpha} \tag{11}$$

$$c_t,\ k_{t+1},\ g_{t+1} \geq 0 \tag{12}$$

$$k_0, g_0 > 0 \text{ given}$$

Now we want to characterize the solution. Let λ be the multiplier on (10), μ_t be the multiplier on (11), and note that the non-negativity constraints will not be binding. The lagrangian of this function is

$$\mathcal{L} = \sum_{t=0}^{\infty} \Big(\beta^t \big[\frac{c_t^{1-\sigma}}{1-\sigma} + \lambda c_t^{-\sigma}((1-\alpha)Ak_t^{\alpha}g_t^{1-\alpha} - c_t)\big] + \mu_t(Ak_t^{\alpha}g_t^{1-\alpha} - c_t - k_{t+1} - g_{t+1})\Big)$$

For notational ease, I will define

$$W(c_t,k_t,g_t) = c_t^{1-\sigma}(\frac{1}{1-\sigma} - \lambda) + \lambda c_t^{-\sigma}(1-\alpha)Ak_t^{\alpha}g_t^{1-\alpha}$$

$$F(k_t,g_t) = Ak_t^{\alpha}g_t^{1-\alpha}$$

Then the first order conditions are

$$\beta^t W_c(t) = \mu_t \tag{13}$$

$$\beta^t W_k(t) + \mu_t F_k(t) = \mu_{t-1} \tag{14}$$

$$\beta^t W_g(t) + \mu_t F_g(t) = \mu_{t-1} \tag{15}$$

With these first order conditions, the resource constraint, the transversality condition, and the initial conditions we can characterize the solution to the Ramsey Problem.

1.4 Deriving Optimal Taxation Results

At the heart of Ramsey's method is deriving results about which taxation schemes might be optimal. Once we have our allocations from section 1.3, we can go back to the TDCE and ask which taxes implement those allocations. One constructive example is to ask how to implement the steady state solution of the Ramsey problem in the limit. Thus, we want to characterize the solution to the problem when

$$\begin{aligned} c_t &\to c_{rp} \\ k_{t+1} &\to k_{rp} \\ g_{t+1} &\to g_{rp} \end{aligned}$$

In order to derive these solutions, it is useful to characterize the partial derivatives of W in equations (13) through (15).

$$\begin{aligned} W_c(t) &= c_t^{-\sigma}(1-\lambda(1-\sigma)) - \lambda\sigma c_t^{-\sigma-1}(1-\alpha)Ak_t^{\alpha}g_t^{1-\alpha} \\ W_k(t) &= \lambda c_t^{-\sigma}(1-\alpha)\alpha Ak_t^{\alpha-1}g_t^{1-\alpha} \\ W_g(t) &= \lambda c_t^{-\sigma}(1-\alpha)(1-\alpha)Ak_t^{\alpha}g_t^{-\alpha} \end{aligned}$$

First, we can use (15) and (14) to characterize the optimal g_t relative to the private capital stock.

$$\begin{aligned} \beta^t W_k(t) + \mu_t F_k(t) &= \beta^t W_g(t) + \mu_t F_g(t) \\ \beta^t \lambda c_t^{-\sigma}(1-\alpha)\alpha Ak_t^{\alpha-1}g_t^{1-\alpha} + \mu_t \alpha Ak_t^{\alpha-1}g_t^{1-\alpha} &= \\ \beta^t \lambda c_t^{-\sigma}(1-\alpha)(1-\alpha)Ak_t^{\alpha}g_t^{-\alpha} &+ \mu_t(1-\alpha)Ak_t^{\alpha}g_t^{-\alpha} \\ \alpha Ak_t^{\alpha-1}g_t^{1-\alpha}\left[\beta^t\lambda c_t^{-\sigma}(1-\alpha)+\mu_t\right] &= (1-\alpha)Ak_t^{\alpha}g_t^{-\alpha}\left[\beta^t\lambda c_t^{-\sigma}(1-\alpha)+\mu_t\right] \\ \alpha Ak_t^{\alpha-1}g_t^{1-\alpha} &= (1-\alpha)Ak_t^{\alpha}g_t^{-\alpha} \\ g_t &= \frac{1-\alpha}{\alpha}k_t \end{aligned}$$

So we get the same proportion of private and government capital, which is encouraging. This means that $g_{rp} = \dfrac{1-\alpha}{\alpha}k_{rp}$ Now lets focus on the intertemporal substitution. By dividing (13) across time, we can get an Euler Equation.

$$\frac{W_c(t)}{W_c(t+1)} = \beta\frac{\mu_t}{\mu_{t+1}}$$

From the first order condition for capital, we can get an expression for $\frac{\mu_t}{\mu_{t+1}}$.

$$\begin{aligned} \beta^{t+1}W_k(t+1) + \mu_{t+1}F_k(t+1) &= \mu_t \\ \frac{\beta^{t+1}W_k(t+1)}{\mu_{t+1}} + F_k(t+1) &= \frac{\mu_t}{\mu_{t+1}} \\ \frac{\beta^{t+1}W_k(t+1)}{\beta^{t+1}W_c(t+1)} + F_k(t+1) &= \frac{\mu_t}{\mu_{t+1}} \\ \frac{W_k(t+1)}{W_c(t+1)} + F_k(t+1) &= \frac{\mu_t}{\mu_{t+1}} \end{aligned}$$

Thus, we get

$$\frac{W_c(t)}{W_c(t+1)} = \beta(F_k(t+1) + \frac{W_k(t+1)}{W_c(t+1)})$$

In the steady state, the left hand side of the equation is 1 and we can plug in the functional forms to get

$$1 = \beta\left[\alpha A k_{rp}^{\alpha-1} g_{rp}^{1-\alpha} + \frac{\lambda c_{rp}^{-\sigma}(1-\alpha)\alpha A k_{rp}^{\alpha-1} g_{rp}^{1-\alpha}}{c_{rp}^{-\sigma}(1-\lambda(1-\sigma)) - \lambda\sigma c_{rp}^{-\sigma-1}(1-\alpha)A k_{rp}^{\alpha} g_{rp}^{1-\alpha}}\right]$$

$$1 = \beta\left[\alpha A k_{rp}^{\alpha-1} g_{rp}^{1-\alpha} + \frac{\lambda(1-\alpha)\alpha A k_{rp}^{\alpha-1} g_{rp}^{1-\alpha}}{1-\lambda(1-\sigma) - \lambda\sigma\frac{(1-\alpha)}{c_{rp}} A k_{rp}^{\alpha} g_{rp}^{1-\alpha}}\right]$$

$$1 = \beta\alpha A k_{rp}^{\alpha-1} g_{rp}^{1-\alpha}\left[1 + \frac{\lambda(1-\alpha)}{1-\lambda(1-\sigma) - \lambda\sigma(1-\alpha)\frac{A k_{rp}^{\alpha} g_{rp}^{1-\alpha}}{c_{rp}}}\right]$$

Now we can compare this equation to the steady state of the TDCE environment, from (6),

$$1 = \beta(1-\tau_\infty)\alpha A k_\infty^{\alpha-1} g_\infty^{1-\alpha}$$

Thus, we see that we must have that the taxation in the limit converges to

$$\tau_\infty = \frac{-\lambda(1-\alpha)}{1-\lambda(1-\sigma) - \lambda\sigma(1-\alpha)\frac{A k_{rp}^{\alpha} g_{rp}^{1-\alpha}}{c_{rp}}}$$

This term is a little difficult to interpret, but it is probably not zero. So the typical result from Chamley and Judd that taxation in the result should be equal to zero does not hold in this case. As a sanity check, examine the different parameters and think about the value of tax for $\lambda = 0$ or $\alpha = 1$. What do these values imply? Does it make sense that capital taxation in the limit might be negative?