Economics 8105
Macroeconomic Theory
Recitation 3

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Outline:

- Minnesota Economics Lecture
- Problem Set 1
- Midterm Exam
- Fit Growth Model into SLP
- Corollary of Contraction Mapping Theorem
- Properties of the Value Function

1 Applying SLP to the Single Sector Growth Model

Chapter 4 of Stokey Lucas Prescot demonstrates that we can write a sequential maximization problem as a dynamic program and shows us what kind of assumptions are needed to infer properties about the value function. In these notes, we will apply those methods to the single sector growth model and walk through some of the more fundamental proofs.

SLP focuses on models that can be written as

$$\begin{aligned} v^*(x_0) = \sup_{\{x_{t+1}\}} \quad & \sum_{t=0}^{\infty} \beta^t F\left(x_t, x_{t+1}\right) \\ s.t. \quad & x_{t+1} \in \Gamma\left(x_t\right), \quad \forall t \geq 0 \\ & x_0 \in X \text{ is given} \end{aligned}$$

with the corresponding functional equation,

$$v(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}, \quad \forall x \in X$$

under the assumption that F is bounded and β is less than one. We will show that we can fit the single sector growth model into this format. In the single sector growth model that we talked about last week, we showed that we can write the problem as

$$\begin{aligned} v^*(k_0) = \sup_{\{x_{t+1}\}} \quad & \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \\ s.t. \quad & 0 \leq k_{t+1} \leq f(k_t), \quad \forall t \geq 0 \\ & k_0 \text{ is given.} \end{aligned}$$

Claim 1. *Under relatively mild assumptions (what are they?), u can be expressed as a continuous and bounded function of $k, k' \in X$ where $k \in X$, a convex subset of $\mathbb{R}$ and $k' \in \Gamma(k)$, where $\Gamma : X \to X$ is a non-empty, compact-valued, and continuous correspondence.*

Proof. Question 4 on problem set 2. □

Under these assumptions, and the standard assumption that β is less than one, we can apply the theorems in SLP 4.1 (Principle of Optimality) to show that we can express our sequence problem as the following functional equation,

$$v(k) = \max_{k' \in \Gamma(k)} \{u(f(k) - k') + \beta v(k')\}, \quad \forall k \in X \tag{1}$$

Now we will demonstrate that the functional equation has a unique solution v and establish some useful properties of v. First, define the operator T to be

$$(Tw)(k) = \max_{k' \in \Gamma(k)} \{u(f(k) - k') + \beta w(k')\} \tag{2}$$

such that equation (1) can be written as $Tv = v$. We have already shown that u is bounded. Thus, if w is bounded, then Tw will also be bounded. Therefore we can establish $T : B(X) \to B(X)$ where $B(X)$ is the space of bounded functions. In order to show that T is a contraction, we will apply Blackwell's Sufficient Conditions.

Theorem 2 (Blackwell's Sufficient Conditions)**.** *Let $X \subset \mathbb{R}^l$, and let $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$ with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying*

a *(Monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$ for all $x \in X$ implies $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$.*

b *(Discounting) There exists some $\beta \in (0, 1)$ such that*

$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \text{ all } f \in B(X),\ a \geq 0,\ x \in X.$$

where $(f + a)(x) = f(x) + a$. Then T is a contraction with modulus β.

Claim 3. *The operator $T : B(X) \to B(X)$, as defined in (2) is a contraction.*

Proof. We will show that T is a contraction by showing that it satisfies Blackwell's Sufficient conditions. Note that we have the first part of the hypothesis, $T : B(X) \to B(X)$. We need to show that it satisfies monotonicity and discounting.

(Mono) Consider $v, w \in B(X)$ with $v(k) \geq w(k)\ \forall k \in X$.

$$\begin{aligned}
(Tv)(k) &= \max_{k' \in \Gamma(k)} u(f(k) - k') + \beta v\,(k') \\
(Tv)(k) &\geq \max_{k' \in \Gamma(k)} u(f(k) - k') + \beta w\,(k') \\
(Tv)(k) &\geq (Tw)(k)\ \forall k \in X
\end{aligned}$$

(Disc)

$$\begin{aligned}
T(w + a)(k) &= \max_{k' \in \Gamma(k)} u(f(k) - k') + \beta(w\,(k') + a) \\
&= \max_{k' \in \Gamma(k)} u(f(k) - k') + \beta w\,(k') + \beta a \\
&= T(w)(k) + \beta a
\end{aligned}$$

□

Since T is a contraction, we know that there exists a unique fixed point v such that $Tv = v$, i.e. equation (1) is satisfied.

Proposition 4 (Corollary to the Contraction Mapping Theorem)**.**
Let (S,ρ) be a complete metric space, and let $T : S \to S$ be a contraction mapping with a fixed point $v \in S$. If S' is a closed subset of S, and $T(S') \subseteq S'$, then $v \in S'$. If in addition, $T(S') \subseteq S" \subseteq S'$, then $v \in S"$.

Proof. Take any initial value (in our example a function) $v_0 \in S$, then $\{T^n v_0\}$ is a sequence that converges to v. (This fact is demonstrated in the proof of the contraction mapping theorem). If for an arbitrary $v_0 \in S'$, $Tv_0 \in S'$, then $\{T^n v_0\}$ is a sequence entirely contained in the set S'. If S' is closed, then it must be that v, the limit of the sequence, is also in S'.

Additionally, if for any arbitrary $v_0 \in S'$, $Tv_0 \in S" \subseteq S'$, then it must be true that $v \in S"$. To see this, simply note that we have already shown that $v \in S'$. By our hypothesis, $Tv \in S"$ and we know that $Tv = v$. □

We will use the logic of this corollary to show several important properties. For convenience, I will define $\bar{u}(k, k') \equiv u(f(k) - k')$.

Claim 5. *Under the assumptions made so far, v is a continuous function and the policy correspondence G is non-empty and upper hemi-continuous.*

Proof. Problem 4 on problem set 2. □

Claim 6. *If $\bar{u}(\cdot, k')$ is strictly increasing in k and Γ has the property that for $k_1 > k_2$, $\Gamma(k_1) \supseteq \Gamma(k_2)$ (what conditions guarantee this?), then v is strictly increasing.*

Proof. Consider our contraction operator as defined in (2),

$$(Tw)(k) = \max_{k' \in \Gamma(k)} \{\bar{u}(k, k') + \beta w(k')\}$$

Let $C^{'}(X)$ define the set of weakly increasing functions and let $C^{''}(X)$ describe the set of strictly increasing function. Note that $B(X) \supseteq C^{'}(X) \supseteq C^{''}(X)$, and that $C^{'}(X)$ is a closed set. From the Corollary to the contraction mapping theorem, we only need to show that the set of weakly increasing functions maps into the set of strictly increasing functions.

Let w be an arbitrary weakly increasing function of k, and consider $k_1, \ k_2$ such that $k_1 > k_2$.

$$\begin{aligned}
(Tw)(k_1) &= \max_{k' \in \Gamma(k_1)} \bar{u}(k_1, k') + \beta w(k') \\
&> \max_{k' \in \Gamma(k_1)} \bar{u}(k_2, k') + \beta w(k') \\
&> \max_{k' \in \Gamma(k_2)} \bar{u}(k_2, k') + \beta w(k') \\
(Tw)(k_1) &> (Tw)(k_2)
\end{aligned}$$

We started with an arbitrary weakly increasing function and showed that Tw is a strictly increasing function. Therefore, we have proven that the fixed point v must be strictly increasing. □

In this next proof we will need $\bar{u}$ to be strictly concave: for any two points, (k_1, k_1') and (k_2, k_2') where $k_i \in \Gamma(k_i')$,

$$\bar{u}(\lambda k_1 + (1-\lambda)k_2, \lambda k_1' + (1-\lambda)k_2') > \lambda\bar{u}(k_1, k_1') + (1-\lambda)\bar{u}(k_2, k_2')$$

or

$$u(f(\lambda k_1 + (1-\lambda)k_2) - (\lambda k_1' + (1-\lambda)k_2')) > \lambda u(f(k_1) - k_1') + (1-\lambda)u(f(k_2) - k_2').$$

We also need Γ to have a convex graph: for any two points, (k_1, k_1'), (k_2, k_2') where $k_i' \in \Gamma(k_i)$,

$$\lambda k_1' + (1-\lambda)k_2' \in \Gamma(\lambda k_1 + (1-\lambda)k_2)$$

What conditions do we have to assume on u and f in order for these properties to be satisfied?

Claim 7. *If $\bar{u}(\cdot, k')$ is strictly concave and Γ has a convex graph, then v is strictly concave.*

Proof. Consider our contraction operator as defined in (2),

$$(Tw)(k) = \max_{k' \in \Gamma(k)} \{\bar{u}(k, k') + \beta w(k')\}$$

Let $C^{'}(X)$ define the set of weakly concave functions and let $C^{''}(X)$ describe the set of strictly concave function. Note that $B(X) \supseteq C^{'}(X) \supseteq C^{''}(X)$, and that $C^{'}(X)$ is a closed set. From the Corollary to the contraction mapping theorem, we only need to show that the set of weakly concave functions maps into the set of strictly concave functions.

Let w be an arbitrary concave function, and consider some $k_1,\ k_2$ with $k_1 \neq k_2$ and $\lambda \in (0,1)$. First, I will define $k_1' \in \Gamma(k_1)$ and $k_2' \in \Gamma(k_2)$ as the optimal choices given each starting level of capital.

$$\begin{aligned}
(Tw)(\lambda k_1 + (1-\lambda)k_2) &= \max_{k' \in \Gamma(\lambda k_1 + (1-\lambda)k_2)} \bar{u}(\lambda k_1 + (1-\lambda)k_2, k') + \beta w(k') \\
&\geq \bar{u}(\lambda k_1 + (1-\lambda)k_2, \lambda k_1' + (1-\lambda)k_2') + \beta w(\lambda k_1' + (1-\lambda)k_2') \\
&> \lambda\bar{u}(k_1, k_1') + (1-\lambda)\bar{u}(k_2, k_2') + \beta[\lambda w(k_1') + (1-\lambda)w(k_2')] \\
(Tw)(\lambda k_1 + (1-\lambda)k_2) &> \lambda(Tw)(k_1) + (1-\lambda)(Tw)(k_2)
\end{aligned}$$

We started with an arbitrary weakly increasing function and showed that Tw is a strictly increasing function. Therefore, we have proven that the fixed point v must be strictly increasing. $\square$

2 Summarizing Properties of the Value Function

The following is a helpful table assembled by Monica Tran Xuan to summarize the assumptions and theorems in section 4.2 of SLP. We start with assumptions 4.3 and 4.4 so that there is a unique solution v to the (FE).

Assumptions	**Theorems**	**Properties of** v	**Properties of** G
A4.5: $F(\cdot, y)$ is strictly increasing A4.6: Γ is monotone. i.e. $x \leq x' \Rightarrow \Gamma(x) \subseteq \Gamma(x')$	Thm 4.7	v is strictly inceasing	
A4.7: F is concave in (x, y) A4.8: Γ is convex, i.e. $\forall \lambda \in [0,1], y \in \Gamma(x), y' \in \Gamma(x')$ $\Rightarrow \lambda y + (1-\lambda)y' \in \Gamma(\lambda x + (1-\lambda)x')$	Thm 4.8	v is strictly concave	G is single-valued, continuous function
A4.7: F is concave in (x, y) A4.8: Γ is convex A4.9: F is continuosly differentiable on the interior of A (w.r.t. x)	Thm 4.11	v is continuously differentiable at $x_0 \in int(X)$	