Economics 8105
Macroeconomic Theory
Recitation 2

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Outline:

- Transversality Conditions
- Pareto Efficiency/Optimality
- Welfare Theorems
- Social Planner's Problem
- Introduction to Dynamic Programming

1 Pareto Optimality and the Welfare Theorems

In this section, I present the following definitions and theorems for general production economy similar to the one presented in week 1.

Definition 1.1. An allocation $z = \left\{ \left(z^{H,i}\right)_{i \in I}, z^F \right\}$ is feasible if $\forall i \in I,\ z^{H,i} \in X^i$, $z^F \in Y$, and z satisfies markets clearance.

Definition 1.2. An allocation z is Pareto Optimal if it is feasible, and there exists no other feasible allocation $\hat{z}$ such that

$$\forall i \in I,\ U^i\left(\left\{(\hat{c}_t^i, \hat{l}_t^i)\right\}_{t=0}^{\infty}\right) \geq U^i\left(\left\{(c_t^i, l_t^i)\right\}_{t=0}^{\infty}\right)$$
$$\exists j \in I : U^j\left(\left\{(\hat{c}_t^j, \hat{l}_t^j)\right\}_{t=0}^{\infty}\right) > U^j\left(\left\{(c_t^j, l_t^j)\right\}_{t=0}^{\infty}\right)$$

Theorem 1.3. ***First Welfare Theorem***

Let $\mathcal{E}$ be a production economy such that $\forall i$, $k_0^i > 0$ and U^i is strictly increasing. If (z, p) is an Arrow-Debreu (competitive) equilibrium (where $z = \left\{ \left(z^{H,i}\right)_{i \in I}, z^F \right\}$), then z is Pareto optimal.

Proof. Suppose, by contradiction, that z is not Pareto optimal. Then by definition, there exists another feasible allocation $\hat{z}$ such that

$$\forall i \in I,\, U^i\left(\left\{(\hat{c}_t^i, \hat{l}_t^i)\right\}_{t=0}^{\infty}\right) \geq U^i\left(\left\{(c_t^i, l_t^i)\right\}_{t=0}^{\infty}\right)$$
$$\exists j \in I : U^j\left(\left\{(\hat{c}_t^j, \hat{l}_t^j)\right\}_{t=0}^{\infty}\right) > U^j\left(\left\{(c_t^j, l_t^j)\right\}_{t=0}^{\infty}\right)$$

Claim-1 $\sum_{t=0}^{\infty} p_t \left[\hat{c}_t^j + \hat{x}_t^j\right] > \sum_{t=0}^{\infty} \left[w_t \hat{n}_t^j + r_t \hat{k}_t^j\right] + \pi^j$

Suppose not, i.e. $\sum_{t=0}^{\infty} p_t \left[\hat{c}_t^j + \hat{x}_t^j\right] \leq \sum_{t=0}^{\infty} \left[w_t \hat{n}_t^j + r_t \hat{k}_t^j\right] + \pi^j$

Then we have that $\hat{z}^{H,j}$ satisfies budget constraint and yields higher utility, which contradicts $z^{H,j}$ being part of the Arrow-Debreu equilibrium.

Claim-2 $\forall i$, $\sum_{t=0}^{\infty} p_t \left[\hat{c}_t^i + \hat{x}_t^i\right] \geq \sum_{t=0}^{\infty} \left[w_t \hat{n}_t^i + r_t \hat{k}_t^i\right] + \pi^i$

Suppose not, i.e. $\sum_{t=0}^{\infty} p_t \left[\hat{c}_t^i + \hat{x}_t^i\right] < \sum_{t=0}^{\infty} \left[w_t \hat{n}_t^i + r_t \hat{k}_t^i\right] + \pi^i$

Then there exists $\epsilon > 0$ such that $\sum_{t=0}^{\infty} p_t \left[\hat{c}_t^i + \hat{x}_t^i\right] + \epsilon \leq \sum_{t=0}^{\infty} \left[w_t \hat{n}_t^i + r_t \hat{k}_t^i\right] + \pi^i$

Define a new allocation for HH i , $\tilde{z}^{H,i} = \{(\tilde{c}_t^i, \hat{l}_t^i, \hat{n}_t^i, \hat{k}_t^i, \hat{x}_t^i)\}_{t=0}^{\infty}$ such that $\tilde{c}_0^i = \hat{c}_t^i + \frac{\epsilon}{p_0}$ and $\forall t \geq 1$, $\tilde{c}_t^i = \hat{c}_t^i$

Then $\sum_{t=0}^{\infty} p_t \left[\tilde{c}_t^i + \tilde{x}_t^i\right] = \sum_{t=0}^{\infty} p_t \left[\hat{c}_t^i + \hat{x}_t^i\right] + \epsilon \leq \sum_{t=0}^{\infty} \left[w_t \hat{n}_t^i + r_t \hat{k}_t^i\right] + \pi^i$

Then we have that $\tilde{z}^{H,i}$ satisfies the same budget constraint as $\hat{z}^{H,i}$. However, since U^i is strictly increasing,

$$U^i\left(\left\{(\tilde{c}_t^i, \tilde{l}_t^i)\right\}_{t=0}^{\infty}\right) > U^i\left(\left\{(\hat{c}_t^i, \hat{l}_t^i)\right\}_{t=0}^{\infty}\right) \geq U^i\left(\left\{(c_t^i, l_t^i)\right\}_{t=0}^{\infty}\right)$$

which contradicts $z^{H,j}$ being part of the Arrow-Debreu equilibrium.

Summing across all HHs, we have

$$\sum_{i \in I}\left[\sum_{t=0}^{\infty} p_t \left[\hat{c}_t^i + \hat{x}_t^i\right]\right] > \sum_{i \in I}\left[\sum_{t=0}^{\infty} \left[w_t \hat{n}_t^i + r_t \hat{k}_t^i\right]\right] + \sum_{i \in I} \pi^i$$

Equivalently,

$$\sum_{t=0}^{\infty} p_t \left[\sum_{i\in I} \hat{c}_t^i + \sum_{i\in I} \hat{x}_t^i\right] > \sum_{t=0}^{\infty} \left[w_t \sum_{i\in I} \hat{n}_t^i + r_t \sum_{i\in I} \hat{k}_t^i\right] + \sum_{i\in I} \pi^i$$

Note that

$$\sum_{i\in I} \pi^i = \sum_{i\in I} \theta^i \sum_{t=0}^{\infty} \left[p_t y_t^f - w_t n_t^f - r_t k_t^f\right] = \sum_{t=0}^{\infty} \left[p_t y_t^f - w_t n_t^f - r_t k_t^f\right]$$

$$\sum_{t=0}^{\infty} \left[p_t y_t^f - w_t n_t^f - r_t k_t^f\right] \geq \sum_{t=0}^{\infty} \left[p_t \hat{y}_t^f - w_t \hat{n}_t^f - r_t \hat{k}_t^f\right]$$

Subtituting in the profit condition and the firm's problem, we have

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t \left[\sum_{i\in I} \hat{c}_t^i + \sum_{i\in I} \hat{x}_t^i\right] &> \sum_{t=0}^{\infty} \left[w_t \hat{n}_t^f + r_t \hat{k}_t^f\right] + \sum_{t=0}^{\infty} \left[p_t y_t^f - w_t n_t^f - r_t k_t^f\right] \\
&\geq \sum_{t=0}^{\infty} \left[w_t \hat{n}_t^f + r_t \hat{k}_t^f\right] + \sum_{t=0}^{\infty} \left[p_t \hat{y}_t^f - w_t \hat{n}_t^f - r_t \hat{k}_t^f\right] \\
&= \sum_{t=0}^{\infty} p_t \hat{y}_t^f
\end{aligned}$$

Thus, we have

$$\sum_{t=0}^{\infty} p_t \left[\sum_{i\in I} \hat{c}_t^i + \sum_{i\in I} \hat{x}_t^i\right] > \sum_{t=0}^{\infty} p_t \hat{y}_t^f$$

Note that by contradiction hypothesis, $\hat{z}$ is feasible so $\forall t$

$$\sum_{i\in I} \hat{c}_t^i + \sum_{i\in I} \hat{x}_t^i \leq y_t^f$$

Multiplying both sides by p_t and summing across of time, we have

$$\sum_{t=0}^{\infty} p_t \left[\sum_{i\in I} \hat{c}_t^i + \sum_{i\in I} \hat{x}_t^i\right] \leq \sum_{t=0}^{\infty} p_t y_t^f$$

which is a contradiction. $\square$

Theorem 1.4. ***Second Welfare Theorem***

Let $\mathcal{E}$ be a production economy such that $\forall i, k_0^i > 0$ and $\forall i$, U^i is continuous, strictly increasing, and concave. F is continuous, increasing, and concave. If $z = \left\{ \left(z^{H,i}\right)_{i \in I}, z^F \right\}$ is a Pareto optimal allocation, then there exists a price system p and reallocation endownments $\left(k_0^{i'}, \theta^{i'}\right)$ such that (z, p) is an Arrow-Debreu equilibrium of $\mathcal{E}'$, the economy defined by new endowments.

Theorem 1.5. ***Second Welfare Theorem (Version with Transfers)***

Let $\mathcal{E}$ be a production economy such that $\forall i, k_0^i > 0$ and $\forall i$, U^i is continuous, strictly increasing, and concave. F is continuous, increasing, and concave. If $z = \left\{ \left(z^{H,i}\right)_{i \in I}, z^F \right\}$ is a Pareto optimal allocation, then there exists a price system p and transfers $\left(T^i\right)_{i \in I}$ such that (z, p) is an Arrow-Debreu equilibrium with these transfers.

2 Environment

- Discrete time $t = 0, 1, 2, \ldots$
- Production economy with one commodity
- HHs:
  - One infinitely lived, representative consumer
  - Utility function: $U\left(\{c_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t u(c_t)$
  - The Utility function is strictly increasing, differentiable, strictly concave, and satisfies the Inada conditions.
  - Consumer invests x_t
  - The Consumer supplies 1 unit of labor inelastically in every period
  - Consumer has capital stock k_t which fully depreciates every period
  - Law of Motion of Capital: $k_{t+1} \leq x_t$
  - Consumer begins with initial capital k_0
  - Consumer rents out capital services to firms, receiving capital income.
  - Consumers own a share of firm profits, but profits will be zero.
- Firms: only 1 sector producing goods that can either be consumed or invested
  - One representative firm.
  - Final good is produced by: $y_t^f = F(k_t^f, n_t^f)$

- F is increasing, strictly concave, and homogeneous of degree one.
- Let $f(k_t^f) = F(k_t^f, 1)$

Definition 2.1. An **Arrow-Debreu Equilibrium** is

- an allocation for the HH: $z^H = \{(c_t, k_t, x_t)\}_{t=0}^{\infty}$
- an allocation for the firm: $z^F = \{(y_t^f, k_t^f, n_t^f)\}_{t=0}^{\infty}$
- a system of prices: $p = \{(p_t, w_t, r_t)\}_{t=0}^{\infty}$

such that

(HH) Given p, z^H solves

$$\begin{aligned} \max_{\{(c_t,k_t,x_t)\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ s.t. & \\ \sum_{t=0}^{\infty} p_t \left[c_t + x_t\right] &\leq \sum_{t=0}^{\infty} \left[w_t + r_t k_t\right] \\ k_{t+1} &\leq x_t, \ \forall t \\ c_t, k_{t+1} &\geq 0, \ \forall t \\ k_0 &> 0, \quad \text{given} \end{aligned}$$

(Firm) Given p, z^F solves

$$\begin{aligned} \max_{\{(y_t^f,k_t^f,n_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} & \left[p_t y_t^f - w_t n_t^f - r_t k_t^f\right] \\ s.t. & \\ y_t^f &\leq F(k_t^f, n_t^f), \ \forall t \\ k_t^f, n_t^f, y_t^f &\geq 0, \forall t \end{aligned}$$

(Mkt) For all t,

$$\begin{aligned} &(\text{Goods Market}) c_t + x_t = y_t^f \leq F(k_t^f, n_t^f) \\ &(\text{Labor Market}) 1 = n_t^f \\ &(\text{Capital Market}) k_t = k_t^f \end{aligned}$$

3 Transversality Condition

The transversality condition is part of the traditional method to characterize the equilibrium allocation. There are multiple ways to interpret this condition. One way is to of it as an optimality condition, i.e. Euler equation (deriving from FOCs) and transversality conditions determine the optimal equilibrium path.

From the households first-order conditions, we can derive the Euler equation which relates the intertemporal marginal rate of substitution of consumption

$$\frac{u'(c_t)}{u'(c_{t+1})} = G(k_t, k_{t+1})$$

Moreover, we can use resource constraint to rewrite this equation into the system of second-order difference equations in (k_t, k_{t+1}, k_{t+2}).

$$\frac{u'(f(k_t) - k_{t+1})}{u'(f(k_{t+1}) - k_{t+2})} = G(k_t, k_{t+1})$$

To solve the system, we need two boundary conditions. The first condition will be the initial condition $k_0^i > 0$ given, and the second condition will be the transversality condition.

Finite Time: Suppose that consumers only live up to period T. What will k_{T+1} be in equilibrium. It must be zero, as the consumer has no use for capital after death.

Infinitely Lived Agents: What should we expect k_{T+1} to be as $T \to \infty$? Naturally, we should expect that it remain at or near zero. A basic formulation of the transversality condition is $\lim_{t\to\infty} p_t k_{t+1} = 0$ where p_t is the Arrow-Debreu prices on consumption. This means that at the limit, capital in the following period has no value relative to time 0 consumption. You can rewrite this equation in terms of k_t and k_{t+1}.

Question. *What are the transversality conditions the sequential markets setting? Hint: there is one for k_{t+1} and one for b_{t+1}.*

4 Social Planner's Problem

4.1 An Environment with n Consumers

The Welfare Theorems establish some sense of equivalence between the competitive equilibrium allocations and Pareto optimal allocations. To find any Pareto optimal allocation, we can solve the problem of a social planner who maximizes weighted sum of consumers' utilities subject to resource constraints:

$$
\begin{aligned}
\max \sum_{i \in I} \alpha^i U^i \left(\left\{ \left(c_t^i, l_t^i \right) \right\}_{t=0}^{\infty} \right) \\
s.t. \\
\sum_{i \in I} \left[c_t^i + x_t^i \right] &= F\left(\sum_{i \in I} k_t^i, \sum_{i \in I} n_t^i\right) \\
k_{t+1}^i &\leq x_t^i + (1-\delta) k_t^i, \ \forall t \\
l_t^i + n_t^i &\leq 1, \ \forall t \\
c_t^i, k_{t+1}^i, l_t^i, n_t^i &\geq 0, \ \forall t \\
k_0^i &> 0 \text{ given}
\end{aligned}
$$

Using methods such as the Negishi Method, we can then find the pareto optimal allocations that correspond to competitive equilibria.

4.2 In Our Representative Agent Environment

In the environment characterized in section 2, the social planner's problem is equivalent to the competitive equilibrium. This can be shown explicitly by characterizing equilibria in each setting with first order conditions, resource constraints, and the transversality condition. The planner's problem simplifies to

$$
\begin{aligned}
\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
s.t. \\
c_t + k_{t+1} &= f(k_t) \\
c_t, k_{t+1} &\geq 0, \ \forall t \\
k_0 &> 0 \text{ given}
\end{aligned}
$$

This is a traditional neoclassical growth model.

5 Introduction to Dynamic Programming

5.1 The Finite Horizon Case

Suppose the representative agent in the neoclassical growth model that we have developed lives only T periods:

$$\begin{aligned} \max_{c_t, k_{t+1}} \quad & \sum_{t=0}^{T} \beta^t u(c_t) \\ s.t. \quad & c_t + k_{t+1} \leq f(k_t), \\ & c_t, k_{t+1} \geq 0, \qquad t = 0, 1, \ldots, T, \\ & \text{given } k_0. \end{aligned}$$

Under the assumptions made in section 2 the Euler equations, the transversality condition, and the resource constraints are sufficient to characterize the optimal allocations. Even so, it is not trivial to solve the problem analytically. We can simplify our task if we can write the problem recursively.

Question. *How do we write this problem recursively?*

Let's work backwards from the final period. Given the fact that the social planner has already chosen the capital stock in T to be k_T, we can find the allocation in the final period of the agents life by maximizing the one period problem:

$$\begin{aligned} v_T^*(k_T) = \max_{c_T, k_{T+1}} \quad & u(c_T) \\ s.t. \quad & c_T + k_{T+1} \leq f(k_T), \\ & c_T, k_{T+1} \geq 0. \end{aligned}$$

Note that in micro theory, $v(\cdot)$ is the indirect utility of the agent. As we argued earlier, the optimal choice for the planner is to set $k_{T+1} = 0$. Then, we will have:

$$v_T^*(k_T) = u(f(k_T)).$$

What about the prior period? Given the fact that optimal choice of capital stock in T is to set $k_{T+1} = 0$, the value of the planning problem in $T - 1$ becomes:

$$\begin{aligned} v_{T-1}^*(k_{T-1}) = \max_{c_{T-1}, k_T} \quad & u(c_{T-1}) + \beta u(f(k_T)) \\ s.t. \quad & c_{T-1} + k_T \leq f(k_{T-1}), \\ & c_{T-1}, k_T \geq 0. \end{aligned}$$

If we substitute for $u\left(f\left(k_T\right)\right)$, we can write this as:

$$\begin{aligned} v_{T-1}^*\left(k_{T-1}\right) = \max_{c_{T-1}, k_T} \quad & u\left(c_{T-1}\right) + \beta v_T^*\left(k_T\right) \\ s.t. \quad & c_{T-1} + k_T \leq f\left(k_{T-1}\right), \\ & c_{T-1}, k_T \geq 0. \end{aligned}$$

As long as u has a nice functional form, we could find the solution to this problem with pen and paper.

Let us go one step further back; the value of the planner becomes:

$$\begin{aligned} v_{T-2}^*\left(k_{T-2}\right) = \max_{c_{T-2}, c_{T-1}, k_{T-1}, k_T} \quad & u\left(c_{T-2}\right) + \beta u\left(c_{T-1}\right) + \beta^2 u\left(f\left(k_T\right)\right) \\ s.t. \quad & c_{T-2} + k_{T-1} \leq f\left(k_{T-2}\right), \\ & c_{T-1} + k_T \leq f\left(k_{T-1}\right), \\ & c_{T-2}, k_{T-1}, c_{T-1}, k_T \geq 0. \end{aligned}$$

Since we already solved for $v_{T-1}^*\left(k_{T-1}\right)$, we can use the same trick as before and simplify to a problem which has an attainable analytical solution:

$$\begin{aligned} v_{T-2}^*\left(k_{T-2}\right) = \max_{c_{T-2}, k_{T-1}} \quad & u\left(c_{T-2}\right) + \beta v_{T-1}^*\left(k_{T-1}\right) \\ s.t. \quad & c_{T-2} + k_{T-1} \leq f\left(k_{T-2}\right), \\ & c_{T-2}, k_{T-1} \geq 0. \end{aligned}$$

Therefore, we may conclude that, at any period $0 \leq t \leq T$, the value of the planning problem can be written as:

$$\begin{aligned} v_t^*\left(k_t\right) = \max_{c_t, k_{t+1}} \quad & u\left(c_t\right) + \beta v_{t+1}^*\left(k_{t+1}\right) \\ s.t. \quad & c_t + k_{t+1} \leq f\left(k_t\right), \\ & c_t, k_{t+1} \geq 0. \end{aligned}$$

Given a value function $v_{t+1}^*\left(\cdot\right)$, solving this problem to find $v_t^*\left(\cdot\right)$ is straightforward, and we know the solution for $v_T^*(\cdot)$. Therefore, through backward induction, we can solve for the whole sequence of consumption, capital, and indirect utility.
Question. *What is the interpretation of this problem?*

Suppose I ensure you there is a social planner that will act optimally tomorrow, based on whatever capital stock you provide to her. And you know that you will receive utility from that capital stock according to $v_{t+1}^*\left(\cdot\right)$. Now, the only think you have to decide is what portion of your production today you will consume and what portion you would invest as capital for tomorrow.

Question. *Will $v_t^*(\cdot)$ and $v_{t+1}^*(\cdot)$ be the same in general?*

A: No! One simple reason for this is that, at date t, $T-t$ periods of agent's life is remaining, while at $t+1$, $T-t-1$!

The first order conditions also provide solutions for the optimal values of consumption and investment as functions of k_t. Let us denote these functions by $g_{k,t}^*(\cdot)$ and $g_{c,t}^*(\cdot)$, and call them *policy functions*; since these give the optimal policy for the planner to take, given the *current situation* in the economy, which is given by the capital stock. Therefore:

$$\begin{aligned} c_t^* &= g_{c,t}^*(k_t), \\ k_{t+1}^* &= g_{k,t}^*(k_t). \end{aligned}$$

5.2 Bellman's Equation

Consider the recursive formula for the social planner's problem for our infinitely lived agent. Because we no longer have a finite number of periods, it would be convenient for v_t^* to be the same $\forall t$. The following functional equation is typically referred to as a bellman equation and characterizes a typical dynamic programming problem.

$$\begin{aligned} v^*(k_t) = \max_{c_t, k_{t+1}} \quad & \{u(c_t) + \beta v^*(k_{t+1})\} \\ s.t. \quad & c_t + k_{t+1} \leq f(k_t), \\ & c_t, k_{t+1} \geq 0. \end{aligned}$$

Under certain conditions, which you will explore on the next problem set, we can apply the contraction mapping theorem to guarantee that such a v^* exists and we can prove the solution to the bellman equation is that same as the solution to the sequential formulation of the planner's problem (SLP Section 4.1).

6 Guess and Verify

Given the Bellman's Equation, in some cases, we can solve for the optimal path analytically by guess and verify.

Step 1: Set up the Bellman Equation

Step 2: Guess a functional form for $V(\cdot)$

Step 3: Now that the Bellman Equation is fully specified, solve for the optimal policy functions: $g_c(\cdot)$, $g_k(\cdot)$, etc

Step 4: Substitute the solved policy functions back into Bellman Equation to solve for the value function, $V(\cdot)$

Step 5: If you can solve for the parameters of the function you guessed, then you have verified your guess.