Macroeconomic Theory (ECON 8105)

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Fall 2016

Problem Set 2

Due Date: September 29th, 2016

Please hand in one physical copy per group and write the names of your group members on the first page.

Question 1: Math Review

a Define a Metric Space. Solve Exercise 3.3(a,b,c) in SLP.

b Define a Normed Vector Space. Solve SLP 3.4 (a,d,e,f).

c Define upper hemi-continuity (uhc) and lower hemi-continuity (lhc).

d Prove that a function is continuous if and only if it is a (single-valued) UHC correspondence.

e Prove that a function is continuous if and only if it is a (single-valued) LHC correspondence.

f Solve SLP 3.13

Question 2: Theorem of the Maximum

Consider the feasibility correspondence:

$$\Gamma(k) = \{k' \in \mathbb{R} |\ 0 \leq k' \leq \theta k^{\alpha}\}$$

a What does it mean for $\Gamma(\cdot)$ to be continuous? Prove that $\Gamma(\cdot)$ is continuous.

b State the Theorem of the Maximum. Now, suppose $f(\cdot)$ is a continuous real valued function. What can be said about $g : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$g(k) = \arg\max_{x} f(x) \quad \text{s.t. } x \in \Gamma(k).$$

c Suppose now that $f(\cdot)$ is strictly quasi-concave. What can be said about g?

Question 3: Blackwell's Sufficient Conditions and The Contraction Mapping Theorem

Let $X \subset \mathbb{R}^l$ and $B(X)$ be the space of bounded functions, $f : X \to \mathbb{R}$ with the *sup* norm.

1. State Blackwell's Sufficient Conditions for an operator $T : B(X) \to B(X)$
2. Define a contraction. Prove that if T satisfies Blackwells conditions, then T is a contraction.

3. Define a fixed point for T. State and prove the Contraction Mapping Theorem.

Question 4: The One Sector Growth Model

This question aims to help you understand how to apply dynamic programming into characterizing solutions of the standard one-sector neoclassical growth model. It is a methodology that everyone should master. In each step, try to add as few conditions as possible to achieve the desired results.

Consider the following one-sector growth model:

$$
\begin{aligned}
v^*(k_0) = \max_{c_t,k_{t+1},x_t,n_t,l_t} & \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \\
s.t. \quad & c_t + x_t \leq f(k_t, n_t) \\
& k_{t+1} \leq x_t + (1-\delta)k_t \\
& l_t + n_t \leq 1 \\
& c_t, k_t, l_t, n_t \geq 0 \\
& k_0 > 0 \text{ given}
\end{aligned}
$$

a Write down the conditions on U and f such that this social planner's problem can be written as a dynamic programming problem. Write the functional equation.

b Write down the conditions on f and δ such that there exists a maximal sustainable level of capital $\bar{K}$, which will be when the consumer consumes nothing and works as much as possible. What equation does $\bar{K}$ satisfy?

c Write down the conditions on U, f, δ, and β such that Assumption 4.3 and Assumption 4.4 (SLP, pp. 78) hold. Show that these conditions imply the assumptions.

d Let v and G be as defined in SLP section 4.2. Prove that, under these conditions, v is a bounded and continuous function. If you use any theorems from chapter 4 of SLP, you must prove them in your own words.

e Write down the extra conditions on U and f such that Assumption 4.5 and Assumption 4.6 (SLP, pp. 80) hold. Show that these conditions imply the assumptions. What can you say about v now? What about G?

f Write down the extra conditions on U and f such that Assumption 4.7 and Assumption 4.8 (SLP, pp. 80) hold. Show that these conditions imply the assumptions. What can you say about v and G now?

g Write down the extra conditions on U and f such that Assumption 4.9 (SLP, pp. 84) holds. Show that these conditions imply the assumption. What can you say about v now?

Question 5: Guess and Verify I - Optimal Growth with Leisure

Consider the social planning problem of choosing sequences $\{(c_t, k_t, n_t, l_t)\}_{t=0}^{\infty}$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t (\log c_t + \gamma \log l_t)$$

s.t.

$$c_t + k_{t+1} \leq \theta k_t^{\alpha} n_t^{1-\alpha}$$
$$l_t + n_t \leq 1$$
$$c_t, k_t, n_t, l_t \geq 0$$
$$k_0 \text{ given}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, and $\theta > 0$.

a Write down the Bellman equation for this problem.

b Guess that the value function $V(k)$ has the form $a_0 + a_1 \log k$. Find the analytical solution for this value function $V(k)$ and the policy functions $g_c(k),\ g_k(k),\ g_l(k),\ g_n(k)$. (Hint: the policy function for labor is constant.)

c Define the Arrow-Debreu Equilibrium for this world. Calculate the Arrow-Debreu Equilibrium by using the policy functions found in part (b).

d Define the Sequential Markets Equilibrium for this world. Calculate the Sequential Markets Equilibrium by using the policy functions found in part (b).

e Suppose now that there are equal populations of 2 types of consumers, with the same discount factor β. They have different utility functions— $logc_t + \gamma^1 \log l_t$ and $logc_t + \gamma^2 \log l_t$. Does the equilibrium allocation for this new economy solve a dynamic programming problem like that in part (a)? Carefully explain why or why not. If it does solve such a problem, write down the Bellman equation.

Question 6: Homothetic-Homogeneous Problem

Consider the following problem:

$$v(k_0) = \max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t.

$$c_t + x_t \leq A k_t$$
$$k_{t+1} \leq x_t + (1-\delta) k_t$$
$$c_t, k_t \geq 0$$
$$k_0 \text{ given}$$

a Characterize the homogeneity properties of the optimal decision rules in initial capital stock, k_0. Specifically, if the initial condition is ηk_0 instead of k_0, where $\eta > 0$, how will the optimal time paths for consumption, labor supply, investment, and capital change? Prove your claims.

b Show that the value function is homogeneous of degree $1 - \sigma$ in the initial capital stock.

Question 7: Guess and Verify II

Consider the following sequence problem:

$$v(k_0) = \max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.}$$

$$c_t + k_{t+1} \leq Ak_t$$

$$c_t, k_t \geq 0$$

$$k_0 \text{ given}$$

a Write this problem as a dynamic programming problem.

b Assue $u(c) = c^{1-\sigma}/(1 - \sigma)$. Write down the Bellman equation. Make a guess for the value function and obtain an analytical expression for $v(\cdot)$.

c Assume $u(c) = -e^{-c}$. Derive $v(\cdot)$.

Question 8: Different Discount Factors (Prelim Spring 2007)

Consider the competitive equilibrium of an Ak economy with two types of agents with equal mass of each. The utility function of type i is given by:

$$\sum_{t=0}^{\infty} \beta_i^t \log c_{i,t}$$

where $0 < \beta_1 < \beta_2 < 1$. Assume that the initial endowments of period 0 capital stock are the same: $k_{1,0} = k_{2,0} > 0$. The aggregate resource contraint is

$$c_t + k_{t+1} \leq Ak_t + (1 - \delta)k_t$$

where c_t denotes period t aggregate consumption, k_t the aggregate capital stock, and $0 < \delta < 1$ the depreciate rate.

a Do the two agent types consume the same amount in period 0? If not, who consumes more? Prove your claims.

b What is $\lim_{t\to\infty} \frac{c_{1t}}{c_{2t}}$ in equilibrium?